Differential Isostasy approach to calculating isostatic equilibrium (compare two neighboring columns at the same time, or a single column at two different times)

Two Rules (above compensation level)

1. the mass is equal in all columns
   \[ \sum_0^i \Delta (\rho h)_i = 0 \]
   
   0 --> i are levels of differing compositions (densities)
   \( \Delta \) indicates the difference between columns
   \( \rho_i \) = density
   \( h_i \) = thickness of each level

2. differences in thicknesses of levels cause differences in the elevation of the surface (expresses conservation of volume)
   \[ \sum_0^i \Delta h_i = \Delta \text{ elevation} \]

Example 1. If eustatic sea level rises by 300 m, how deep will the water be at a location that was on the beach just before the sea level rise?
BEFORE

h

h

h

h

h

h

h

a

h = thickness of level
a = air
H2O = water
sed = sediment
c = crust
l = lithosphere
A = asthenosphere

step 1

1. \[ \sum_{i=0}^{i} \Delta (\rho h) = 0 = \rho_A(h_{A1}-h_{A2}) + \rho_c(h_{c1}-h_{c2}) + \rho_l(h_{l1}-h_{l2}) + \rho_{H2O}(h_{H2O1}-h_{H2O2}) \]

0 = \rho_A(h_{A1}-h_{A2}) + \rho_{H2O}(-h_{H2O2})

2. \[ \Delta E = E_1-E_2 = \sum_{i=0}^{i} \Delta h = (h_{A1}-h_{A2}) + (h_{c1}-h_{c2}) + (h_{l1}-h_{l2}) + (h_{H2O1}-h_{H2O2}) \]

-300 - 0 = (h_{A1}-h_{A2}) - h_{H2O2}

-300 + h_{H2O2} = (h_{A1}-h_{A2})

***by substitution:

0 = \rho_A(-300+h_{H2O2}) - \rho_{H2O}(h_{H2O2})

h_{H2O2} = \frac{300\rho_A}{\rho_A - \rho_{H2O}}

putting in realistic values of density (asthenosphere 3.3 gm/cm³; water 1.0 gm/cm³):

\[ \frac{\rho_A}{\rho_A - \rho_{H2O}} = 1.4 \]

which is an amplification factor

and \( h_{H2O} = 430 \text{ m} \)

The largest sea level rise believed to have occurred during the Phanerozoic (300 m), added 430 m to the depth of water at any point below sea level.
The basic equation of flexure of the lithosphere is (for point load on infinite, thin plate):

\[ D \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} + (\rho_m - \rho_i)gw = q_a(x) = T(x) \rho_f g \] (3)

the terms have distinct significance:

\[ D \frac{d^4 w}{dx^4} \] refers to the internal stresses due to bending about mid-line of plate

\[ P \frac{d^2 w}{dx^2} \] refers to forces applied on the end of the beam (parallel to beam)

\[ (\rho_m - \rho_i)gw \] is the restoring force applied at the bottom of the plate

\[ q_a(x) = T(x) \rho_f g \] -- the stress applied to top of plate ==> this is the "load" that needs isostatic compensation

where
- \( w \) = vertical deflection
- \( x \) = horizontal direction
- \( D \) = flexural rigidity
- \( P \) = horizontal load = "in-plane" force
- \( \rho_m, \rho_i \) = densities of mantle and material that fills in the surface depression (water, air, or sediment)
- \( g \) = gravity
- \( q_a(x) \) = vertically-applied load (distributed)

\[ q_a(x) = (\rho_1 - \rho_2)gt(x) \]

where \( (\rho_1 - \rho_2) \) is the density contrast of the surface load, and \( t \) is the thickness of surface load

the flexural rigidity, \( D \),

\[ D = \frac{Eh^3}{12(1-v^2)} \]

where \( E \) is Young's modulus
- \( v \) = Poisson's ratio.
- \( h \) = elastic thickness.
Subsidence due to a distributed load

Invent a term $D_{\alpha x}$, that is shorthand for equations, depending upon $\alpha$ and $x$: $D_{\alpha x} = e^{-x/\alpha \cos \frac{\alpha}{\alpha}}$

So for a rectangular distributed load on infinite beam, with this geometry:

Our task is to find the deflection ($w$) at any point (c) along the length of the beam. Distances “a” and “b” are measured as absolute values of distances from the borders of the rectangular load to point c. There are three positions of possible points (c) along the beam relative to the load:

1. when point whose deflection is of interest (c) is under the load
   $$w_c = \frac{q}{2k} \left[ (1-e^{-a/\alpha \cos \frac{\alpha}{\alpha}}) + (1-e^{-b/\alpha \cos \frac{\alpha}{\alpha}}) \right]$$
   which can be expressed in shorthand as
   $$w_c = \frac{q}{2k} (2-D_{\alpha a}D_{\alpha b})$$

2. when point c is to the left of the load
   $$w_c = \frac{q}{2k} (D_{\alpha a}D_{\alpha b})$$

3. when point c is to the right of the load,
   $$w_c = -\frac{q}{2k} (D_{\alpha a}D_{\alpha b})$$
   where
\[ \alpha = \left[ \frac{4D}{4\Delta \rho g} \right]^{\frac{1}{4}} \]

the \( \Delta \rho \) refers to the difference between fluid densities above and below bending plate

and where

\[ q = t \left( \frac{\rho_s - \rho_a}{\rho_m - \rho_a} \right) \]

for \( \rho_s - \rho_a \) representing load density contrast (sediment replacing air, for instance), and \( \rho_m - \rho_a \) representing restoring force (density contrast above (air) and below (asthenospheric mantle) plate). [k here specifically represents the restoring force, and q represents the force applied above the plate, e.g., the load]