SEISMIC RAY THEORY IN A SPHERICAL EARTH

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Curved, refracted seismic rays in a spherical earth, but plane waves locally
(Think globally, but act locally!)

The figure above shows a local region of the wave front small enough that the wave front is nearly plane and the ray nearly straight. In this small region, we can define a rectangular coordinate system with local x, y and z axes, where x is along the ray in the direction of propagation and y and z are in the plane of the wave front (in the figure, the y axis is perpendicular to the section). With these local axes, we can derive the simple plane wave relationships developed in Section 1 between stress and strain, wave velocity, wave particle motion, etc. We will also use this approximation below to derive Snell’s law for the seismic rays. Locally, we model refraction and reflection at a boundary as the interaction of a plane wave with a plane boundary.
Variables

Ray theory in a spherically symmetric earth deals with the variation of seismic wave velocities, $V_p$ and $V_s$, as functions of only radius, $R$. Waves from a surface source travel down and refract or reflect back to the surface. We can calculate the paths and travel times of these waves by considering the geometry of the rays which are everywhere perpendicular to the wave fronts. In a spherically symmetric earth, the ray from a particular source to a particular receiver will be located in a vertical plane including the source and receiver points and the center of the earth. This plane gives us a cross sectional view as shown in the figure below. Note that this view is not special to the particular azimuth of the receiver with respect to the source: in a spherically symmetric earth the ray path will be the same for any azimuth about the source. The waves spread out symmetrically from the source, and the only variables of importance are radius, $R$, and angular distance from source to receiver, $\Delta$, defined in the figure on the next page.

The variables of concern here are defined below:

- $D_i$: distance measured in degrees
- $Ro$: ray point
- $Rd$: deepest point
- $Rs$: source at depth $h$
- $h$: epicenter
- $i$: is hypocenter
- $io$: receiver

$R$ without a subscript is the independent radial variable: we seek to determine how seismic wave velocities, density, temperature, pressure and gravity vary as functions of $R$. 
\( R_e = R_0 \) = radius of the equivalent spherical earth (one with the same volume as the real ellipsoidal earth), equal to 6371 km.

\( R_d \) = deepest point (minimum radius) along a particular seismic ray; spherical symmetry requires that the ray on one side of the deepest point is the mirror image of the other side for source and receiver both located on the surface.

\( R_s \) = radius to a source located at some depth, \( h \), beneath the surface: \( R_s + h = R_0 \).

\( \Delta \) = angular distance between two points, measured as the angle between the two radial lines from the center of the earth to the two points; often measured in degrees, but for many calculations radians should be used (remember, calculus likes radians).

\( i \) = angle between the tangent to a ray at some point and the radial line to that point; the ray at a point along the ray is thus characterized by a radius \( R \) and angle \( i \)

\( i_0 \) = the value of \( i \) where the ray reaches or leaves the surface (\( R = R_0 \)); for an incoming ray incident upon the surface, this is often called the angle of incidence.

\( i_s \) = the value of \( i \) where the ray leaves the source. This is the angle that can be used to tag or identify the ray, and it is also one of the essential parameters in determining earthquake focal mechanisms.

**Snell's Law: plane waves reflecting and refracting at plane boundaries**

We now have to go back to the zoomed-in view along a ray, looking at so small a region that the ray is straight and the wavefront plane. At this scale, the internal boundaries of the sphere will also be nearly plane, so we can consider Snell's law in terms of plane waves refracting through or reflecting at plane boundaries, as shown in the following ray diagram.
The diagram for the wavefronts within the box is shown in the next figure, together with Snell's Law.

Snell's Law:

\[
\frac{\sin ip}{Vp_1} = \frac{\sin is}{Vs_1} = \frac{\sin ip}{Vp_2} = \frac{\sin rs}{Vs_2}
\]

The point of intersection of the 5 wavefronts with the boundary travels along the boundary with the apparent velocity of the incident wave. The requirement that the wavefronts remain "stuck together" along the boundary leads to Snell's Law. (the arrows, parallel to the rays, show the direction of travel of the wavefronts)

The transformation of P to S waves (and vice-versa) creates a number of interesting phases in the earth, and also leads to a fundamental separation of wave types. Since P waves convert only to SV polarized S waves (and vice-versa), P and SV type waves are closely coupled. For short wavelength (0.1 - 50 km) body waves this results in complex rays with some legs P and other legs S. A ray once reflected at the surface, for example, might be PP, PS, SP, pP, or sP; the letters indicate the wave types.
of the first leg and second legs, and a capital letter indicates that the wave leaves the source in an downwards direction, while a small letter indicates that the first leg leaves the source in an upwards direction. For longer wave length surface waves, the P-SV coupling produces a special type of surface wave, the Rayleigh wave, while the SH polarized S waves produce Love waves. These two fundamentally different wave types (one has vertical and longitudinal particle motions, the other only transverse horizontal motions) merge respectively into two types of free oscillations of the earth, the spheroidal modes (Rayleigh) and the torsional modes (Love).

We can generalize Snell’s law to the spherically symmetric earth as in the following figure.

Two spherical boundaries separate concentric shells, each with uniform velocities and densities. The location and angles are arbitrary, so the relationship is general.

From the two triangles:

\[ R_2 (\sin i_2') = R_1 (\sin i_1') = B \]

From Snell’s law across plane boundaries:

\[ \frac{\sin i_2'}{\sin i_1'} = \frac{V_1}{V_2} \]

From this result we can generalize as we shrink the layers in thickness and increase their number to the limit of a continuously variable velocity distribution with depth for which the following condition applies all along a specific ray:

\[ \frac{R \sin i}{V} = \text{constant} = \text{Ray Parameter} = P \]
The ray parameter "P" has nothing to do with the name of the wave type, P wave; this is just an unfortunate coincidence. S wave rays also have their ray parameters or P values.

Important specific values of the ray parameter occur near the source, the receiver, and at the deepest point of the ray:

\[ P = \frac{R_s \sin i_s}{V_s} = \frac{R_0 \sin i_0}{V_0} = \frac{R_d \sin (90^\circ)}{V_d} = \frac{R_d}{V_d} \]

were \( R_d \) and \( V_d \) are the radius to the deepest point and velocity at the deepest point, respectively.

The most important thing about the ray parameter P is its simple relationship to the travel time function, \( tt \), which is a function of \( \Delta \). The figure below shows how the derivative of the travel time curve is simply equal to the ray parameter.

Two adjacent rays from the same source are considered. The ray on the right appears at the surface at a slightly greater distance (greater by \( d\Delta \) in angular measure) and arrives at a slightly later time (later by \( dtt \) seconds). The small triangle shown is formed by dropping a perpendicular from the surface point of the left ray to the right ray. The sides of this triangle are the surface segment connecting the rays, the perpendicular just mentioned, and the segment of the right ray between the intersection of the perpendicular and the surface. In the limit as the two rays come closer and closer together, we can consider the triangle to be plane with straight sides and apply simple trigonometry. Note that the interior smaller angle of the triangle is equal to the angle of incidence, \( i_0 \), for the left ray. The sine of this angle, \( \sin i_0 \), is equal to the ratio of the opposite side (the small segment of the second ray) to the hypotenuse (the surface segment). In the limit the length of the small segment of the left ray is just the extra time
required for that ray to propagate compared to the right ray divided by the near surface velocity, or $d\tau(V_0)$. The hypotenuse is simply given by $R_0 d\Delta$ where $R_0$ is the radius of the earth

Note that for calculations $\Delta$ must be measured in radians; if degrees are used, as is commonly done in the location bulletins and other discussions, then the conversion factor $\pi/180$ radians/degree must be appropriately inserted.

Thus simple trigonometry gives

$$\sin i_0 = \frac{(V_0)d\tau}{R_0 d\Delta}$$

This can be rearranged to give the derivative or slope of the travel time curve in terms of the ray parameter $P$ as

$$\frac{d\tau}{d\Delta} = \frac{R_0 \sin i_0}{V_0} = P$$

which is extremely useful, as you will see in lab, and in the following.

The basic problem of seismic ray theory is the determine the seismic wave velocities as functions of depth or radius, i.e. $V_p(R)$ and $V_s(R)$, from observations of travel times as a function of distance, $tt_p(\Delta)$ and $tt_s(\Delta)$. The trial-and-error or forward modeling method uses some assumed $V(R)$ to calculate $tt(\Delta)$, then compare the calculation with the observed $tt(\Delta)$ and then modify the assumptions and try again. Inversion of the observations to calculate $V(R)$ directly from $tt(\Delta)$ is also possible for much (but not all) of the earth. The methods are described below.

**$V(R)$ to $tt(\Delta)$: Forward calculation**

The following diagram shows a tiny piece of a ray. Two radii from the earth's center intersect the ray at two points separated by a distance, $dS$, measured along the ray. As $dS$ is made smaller and smaller, the right triangle shown approaches a plane triangle for which simple trigonometric relationships can be applied. The sides of this triangle include the ray segment of length $dS$; a perpendicular dropped from the point of intersection of the right radius to the left radius, of length $Rd\theta$, where $d\theta$ is the small angle between the two radii; and the segment of the left radius between the perpendicular and the ray. The angle $i$ is as shown in the figure. Following the figure are shown the trigonometric relationships that are the basis for calculating travel time and source-to-station distance for the ray.
These differentials can be integrated to give the travel time, $t_t$, and distance, $\Delta$, given a knowledge of how the velocity, $V$, varies with radius, $R$. The integration is done after first substituting for the variable $i$ by using Snell's Law for a spherically symmetric earth, and thereby introducing the ray parameter $P = (R \sin i)/V$. The resulting integrals give travel time and distance by an integration with respect to the variable $R$. Here $V(R)$ is assumed to be known. This is an example of the "forward" problem in geophysics, where one assumes a model and from this model calculates quantities that can be compared with observations – in this case travel time versus distance, $t_t(\Delta)$. The integrals for this are as follows:

$$ tt = 2 \int_{R_d}^{R_0} \frac{R \, dR}{V^2 \sqrt{R^2 - P^2}} $$

$$ \Delta = 2 P \int_{R_d}^{R_0} \frac{dR}{R \sqrt{R^2 - P^2}} $$

Note that the ray parameter $P$ is a way to tag or identify the ray – it is chosen to begin with – and if we know $V(R)$ we also know $R_d$ for a given ray. For example, we can choose a particular ray leaving the surface with angle $i_0$. Let us assume that we know $V(R)$. The ray parameter for this ray can then be calculated to be $P = (R_0 \sin i_0)/(V(R_0))$. If we know $V(R)$, we can also determine $R/V$ at any depth. At the depth at which $R/V$ is equal to $P$, we have by Snell's law $P = R_d/(V(R_d))$, and from this we can determine $R_d$, and then use this $R_d$ in the integrals to calculate travel time or distance for the chosen ray. Values of $tt$ and $\Delta$ are can thereby calculated for a suite of ray parameters covering the range of rays of interest, and in this way a travel time table with $tt$ values and corresponding values of $\Delta$ can be constructed to give $tt(\Delta)$ for the assumed $V(R)$. This is the strategy that Lab 3 uses: For an
assumed velocity model, systematically choose ray parameters, and for each ray parameter calculate the travel time and the distance.

**tt(Δ) to V(R): Inverse Problem (Herglotz-Weichert Method)**

The "inverse" problem is to calculate \( V(R) \) from measurements of \( tt \) as a function of \( Δ \). We start with values of \( tt(Δ) \) that are well enough determined to yield accurate first derivatives, \( dtt/dΔ \) as a function of \( Δ \). Since the ray parameter is equal to \( dtt/dΔ \), we then have \( P \) as a function of distance, \( P(Δ) \). With much manipulation the relationships can be transformed into a solvable integral equation, (Abel's integral equation), leading to the following solution in terms of integration over a whole family of rays, i.e., an integration over \( Δ \).

\[
\pi \ln \frac{R_0}{R_a} = \int_0^{Δ_a} \cosh^{-1} \left( \frac{P(Δ)}{P_a} \right) dΔ
\]

For a particular ray reaching a distance \( Δ_a \), the integration gives the radius \( R_a \), the radius to the deepest point of the particular ray. The integration is over all the rays that emerge between the source and \( Δ_a \) and is therefore a integration over \( Δ \). We already know \( R_0 \), the radius of the earth. The velocity at \( R_a \), or \( V(R_a) \) can then be calculated with Snell's Law, since we already know \( P \) everywhere along the curve, and, in particular, \( P(Δ_a) = P_a \). Thus

\[
V(R_a) = \frac{R_a}{P_a}
\]

In practice the travel-time curve can be broken into small parts, within each of which the slope \( (P) \) is taken as constant, and the integral thus reduced to a summation. Let the chosen interval of distance for the segments be \( DΔ \). Then

\[
\int_0^{Δ_a} \cosh^{-1} \left( \frac{P(Δ)}{P_a} \right) dΔ = DΔ \sum_{k=1}^{k=a} \cosh^{-1} \left( \frac{P_k}{P_a} \right)
\]

which can then be computed as a sum, given the series of values \( P_k \) for \( k = 1 \) to \( k = a \) (the \( P_k \)'s are the slopes of the little straight segments into which we artificially divide the travel-time curve for the computation).

The calculation must be done successively for increasing "a" from the first to the last segment of the travel-time curve, following the curve along all its branches. For each "a" the summation yields the radius \( R_a \) from which the velocity \( V(R_a) \) can be obtained. We thus start with travel times as a function of distance, calculate the first derivative of this curve as a function of distance, and, from this, calculate radii of the bottoming points of rays penetrating to progressively greater depths as distance is increased. From this we can calculate velocities at those bottoming points, and thereby produce a velocity versus depth profile.
A forbidden region for this technique is a region (such as the outer core) within which no rays from the surface can bottom, i.e. within which no rays from the surface can have their deepest points. At the bottom or deepest point the ray is horizontal. Thus, to have a bottoming point in some region, a ray just leaving the bottoming point in a horizontal direction must refract back to the surface. If the *in situ* velocity decreases with depth, the rays are bent downwards and have a downwards concave curvature. This can happen at the deepest point as long as the downwards curvature is not too great and the ray can still reach the surface again. What is "not too great"? The downwards curvature must not be more that the curvature of a circle with a center at the center of the earth and with a radius to the supposed deepest point of the ray in question. If the ray were the same or more curved than that circle, then the ray would remain at that depth and never reach the surface, or it would penetrate deeper so that we violate the starting assumption that we were at the deepest point.

The case of a discontinuous decrease in velocity (e.g., the lower mantle- outer core structure) is easy to visualize. Imagine a source placed in the outer core at a depth shallower than the deepest point of a ray from the surface which just penetrates the core-mantle boundary. This ray from the surface hits the core-mantle boundary at grazing incidence, and is refracted downwards into the outer core at an angle given by Snell's law. It crosses the outer core with its deepest point below the core-mantle boundary. All other steeper rays from the surface source penetrate the core with bottoming points at greater depths than the grazing ray. Thus no ray from the surface will have its deepest point at a depth in the outer core shallower than the deepest point of the grazing ray, and thus the zone of the outer core between that depth and the core-mantle boundary is a kind of "hidden" zone with respect to surface rays. They pass through it, but do not bottom in it. Since there is no $R_d$ in this zone, the Herglotz-Weichert integral cannot be applied here, and other tricks must be used.

In the flat layered models used in crustal-scale refraction experiments the corresponding phenomenon would be a buried low velocity zone. This layer is not represented on the travel-time curve by a refraction arrival, since no rays can bottom or refract horizontally in that zone.


**Effects of simple velocity increase or decrease with depth**

The figures in the next two pages show the rays and travel time curves for two very simple structures: a discontinuous increase in velocity with depth, and a discontinuous decrease with velocity with depth. These two effects explain much of the complexity in travel time curves in the real earth.
\[
\Delta = 2 \cos^{-1}\left(\frac{R_0}{R_1}\right)
\]

Simple velocity increase: \( V_1 > V_0 \)
\[ \Delta = 2 \cos^{-1}\left(\frac{R_0}{R_1}\right) \]

**Simple velocity decrease:** \( V_1 < V_0 \)

Partial reflection

\[ \frac{\partial t}{\partial \Delta} = \frac{R_0}{V_0} \]

Rays above boundary

Distance, \( \Delta \)

Shadow zone

180° or \( \pi \)

Rays refracted below boundary
**Basic Mantle-Core-Inner Core Structure**

The rays and corresponding travel time curves for the mantle-outer core-inner core structures are shown on the following two pages for a very simple earth model consisting two or three uniform velocity layers. In the next page a high velocity “mantle” overlies a low velocity “core” only, while in the following page the inner core is added. The color coding should make the situation clear.
rays in simple two layer earth model with low velocity core (without inner core) and high velocity mantle

diffracted P

core

mantle

distance, degrees

travel time

P

PKP

diffracted P
rays in outer core: reverse branch

rays in outer core: forward branch

note cusps at c, d and e

rays in three layer earth model including inner core

rays refracted through inner core

rays reflected by inner core
How to see the “Hidden Zone” in the Outer Core

The figure below shows the relevant ray paths for the seismic phases ScS and SKS. Note that for simplicity the inner core is not shown.

We start with travel time curves for ScS and SKS. At a given distance $\Delta_{SKS}$ the travel time is $t_{SKS}$, and the ray parameter (or $dt/d\Delta$) taken at the distance $\Delta_{SKS}$. This ray parameter applies to the S segments in the mantle and the P segment in the outer core. The very same ray parameter also applies to the S wave reflected at the core-mantle boundary, which reaches the surface as ScS with a travel time $t_{ScS}$ at a distance $\Delta_{ScS}$. Thus we can find this position on the ScS travel time curve by matching ray parameters or slopes: we find the position where the slope of the ScS travel time curve is the same as the slope of the SKS curve at $\Delta_{SKS}$. This gives us the travel time and the distance for the S wave segments in the mantle for SKS. We can subtract these from the time and distance for SKS to obtain the time $t_p$ and distance $\Delta_p$ for the P wave segment in the outer core. By doing this at different distances along the SKS and ScS curves, we can obtain a travel time versus distance of P in the outer core as $t_p = t_{SKS} - t_{ScS}$, where the times are taken at points where the slopes of the two curves match, and the corresponding distance $\Delta_p = \Delta_{SKS} - \Delta_{ScS}$ is taken at the same points.