Chapter 12

Solid Mechanics and Structural Geology

Introduction

Up to this point, we have focused mostly on geometry and kinematics, borrowing a smattering of concepts from the much broader realm of solid mechanics. The concepts that we have developed to this point — vectors, tensors, stress, strain, some basic material models, etc. — cover about the first 25-30% of a standard continuum mechanics textbook. This background provides a splendid point of departure for beginning to explore the rich world of solid mechanics. One must become conversant with this world if you want to explore why structures form and behave as they do. Solid mechanics is a broad field and there are entire geology-oriented books devoted to this subject (Johnson, 1970; Turcotte and Schubert, 1982; Jaeger and Cook, 1976; Middleton and Wilcock, 1994; Pollard and Fletcher, 2005).

The purpose of this chapter is to give you a glimpse of the basic approach used in a more complete analysis provided by mechanics, as well as review some fundamental results that are particularly germane to structural geology. Bear in mind that this is only a taste, a teaser for the real thing. Hopefully, this will give you
the motivation to dive in deeper, either on your own or in subsequent classes, to explore this world.

The Mechanical Approach

A mechanical approach involves a clear definition of the components necessary to solve the problem of interest. The starting point is commonly the simplifying assumption that the distribution of properties in a material is continuous; i.e., that the material is a *continuum*. This is the origin of the term, continuum mechanics. Because properties vary smoothly in time or space, we describe them in terms of *gradients*, which mathematically are derivatives. Because the Earth is a three dimensional place, the gradients in which we are interested vary in all directions and are specified along the three axes of our Cartesian coordinate system, so we will define how things vary in terms of *partial differential equations*. We have already had a taste of this in Chapters 7 and 8 where the equations governing three dimensional strain are expressed as partial derivatives of displacement (or po-

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**Physical Principles**
- Conservation of mass
- Conservation of linear momentum
- Conservation of angular momentum
- ± Strain compatibility

Always hold in classical mechanics, regardless of the problem, setting, or materials

**Constitutive Equations**
- Elasticity
- Viscosity
- Plasticity
- Combinations of the above

Depend on the material being analyzed and the environmental conditions (pressure, temperature, time span, etc.)

**Limiting Conditions**
- Boundary conditions
- Initial Values

Depend on the specific problem being analyzed and what we know, a priori, about it

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Figure 12.1 — Hierarchy of components of a full mechanical analysis.
position) with respect to position. The basic approach relies on three levels of abstraction (Fig. 12.1; from most general to most specific): (1) **physical principles**, (2) **constitutive equations**, and (3) **boundary conditions** and **initial values**.

**Physical Principles**

Physical principles are those which apply to any body or substance. Any deformation of a continuous medium that we wish to analyze must conform to these principles which form the fundamental basis of classical mechanics. The first of these is **conservation of mass**. As we have already seen in the introduction of balanced cross-sections and the trishear fault-fold model, the principle of conservation of mass is defined by the **continuity equation**:

\[
\frac{d\rho}{dt} + \rho \frac{\partial(v_i)}{\partial x_i} = 0
\]  

(12.1)

where the first term is the material derivative of density with respect to time (sometimes written using capital “\(D\)”). This equation states that the change in density with time plus the flux of material in or out of the system must be equal to zero. It is from this general equation that we derive the specialized condition of incompressibility for volume constant deformation:

\[
\text{div } \mathbf{v} = \left[ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right] = 0
\]  

(12.2)

The cornerstone of physical principles follows from Newton’s Second Law, which deals with the **conservation of momentum**. The momentum of a body is equal to its mass, \(m\), times its velocity, \(\mathbf{v}\). Newton stated that the rate of change of momentum is proportional to and in the direction of the “impressed” force, \(\mathbf{F}\):

\[
\mathbf{F} = \frac{d(m\mathbf{v})}{dt}
\]  

(12.3a)

The term, “impressed force” means the vector sum of all of the forces acting on a body. If the mass does not vary with time, then we can write Newton’s second law in a more familiar format:
where \( a \) is the acceleration. When the force equals zero, momentum must be constant.

From the condition of Newton’s second law, one can derive the **equations of motion** in various formats (see Malvern, 1969 or Pollard and Fletcher, 2005 for details of the derivation). Perhaps the most general and insightful is given by **Cauchy’s First Law of Motion**:

\[
\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i
\]

As with Equation (12.1), \( d/dt \) is the material time derivative and \( g_i \) is acceleration due to gravity. The terms in this equation have units of force per unit volume. This equation then says that the total force per unit volume is equal to the gradient of stress with distance [with units of \( N \ m^{-2} \ m^{-1} = (kg \ m^{-2}) \ m^{-1} \)] or **surface forces** per volume plus the **body forces** per volume.

**Torque**, or **moment**, is the force multiplied by the distance from a pivot point or fulcrum in a material. In a way exactly analogous to what we have just seen, the **conservation of angular momentum** says that the sum of all torques is equal to the rate of change of total angular momentum. **Cauchy’s Second Law of Motion** is an expression of this and its result is a simple and elegant proof that the stress tensor must be symmetric.

\[
\sigma_{ij} = \sigma_{ji} \quad \text{for} \quad i, j = 1 \text{ to } 3
\]

For bodies in equilibrium, the change of linear and angular momentum with respect to time must equal zero. This condition yields two fundamental relationships: the balance of forces and the balance of moments (i.e., torques):

\[
\sum F = \sum F_{\text{surface}} + \sum F_{\text{body}} = 0 \quad (12.5)
\]

\[
\sum M = 0 \quad (12.6)
\]
These are the starting conditions for analyses involving mechanics of static equilibrium. When you draw a free body diagram, it should depict all of the forces and torques on a body and, if it is a problem in static equilibrium as many problems in geology are, those should all sum to zero.

Finally, for some problems, we wish to ensure that the displacement field associated with a particular strain field is single valued and continuous. That is, the strains imposed produce no gaps or overlaps. This condition of strain compatibility which is specified by St.-Venant’s equations:

\[
\frac{\partial^2 \varepsilon_{ij}}{\partial X_i \partial X_j} + \frac{\partial^2 \varepsilon_{il}}{\partial X_i \partial X_l} - \frac{\partial^2 \varepsilon_{ik}}{\partial X_i \partial X_k} - \frac{\partial^2 \varepsilon_{jl}}{\partial X_j \partial X_l} = 0 \quad (12.7)
\]

where \( \varepsilon_{ij} \) is the infinitesimal strain tensor (Chapter 7). Recall that for infinitesimal strain the material and spatial coordinates are the same. Equation (12.7) represents six equations that must be satisfied if the displacement field is to be smooth and continuous. This equation finds important applications in elasticity theory and is, for example, one of the underlying tenants of the construction of the world strain map (Holt et al., 2000). Nonetheless, in the pantheon of physical laws it is a lesser god and there are a number of perfectly physically plausible geological processes that do not comply.

**Constitutive Equations**

So far in this chapter, we haven’t said anything about materials, yet, and how they respond to applied forces (or body forces). Geological materials are extremely complex and different processes may be active at different scales and even in adjacent mineral grains. Nonetheless, there are a small number of models that successfully describe the macroscopic behavior of many natural materials under different conditions. We have already reviewed the three basic models in Chapter 9: elastic, plastic, and viscous.

**Elasticity**

Because rocks in the upper crust deform by fracturing at even modest strains, elasticity theory is intimately related to the concepts of infinitesimal strain (Chapter
7). We have already seen some of the basic equations of elasticity in Chapter 9. In
elastic deformation, by *Hooke’s Law* stress is linearly related to strain by a variety
of elastic moduli depending on the type of deformation:

\[
\begin{align*}
\sigma_{11} &= E\varepsilon_{11} \quad \text{where } G = \text{Youngs Modulus (for axial deformations)} \\
\sigma_{ij} &= G\left(2\varepsilon_{ij}\right) \quad \text{where } G = \text{Shear Modulus (for } i \neq j; \text{i.e., simple shear deformations)} \\
\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)/3 &= B\varepsilon_{ii} \quad \text{where } B = \text{Bulk Modulus (for volume change deformations)}
\end{align*}
\]  

(12.8)

An important additional parameter, *Poisson’s Ratio*, \(\nu\), defines the relation of the
axial to the transverse strain (Fig. 9.1):

\[
\nu = -\frac{\varepsilon_x}{\varepsilon_y} = -\frac{\left(w_f - w_i\right)}{\left(\ell_f - \ell_i\right)}
\]

(12.9)

For incompressible deformation, \(\nu = 0.5\), but rocks have \(0.10 \leq \nu \leq 0.33\). Poisson’s
ratio can be used to relate the elastic moduli of equation (12.8):

\[
G = \frac{E}{2(1+\nu)} = \frac{3B(1-2\nu)}{2(1+\nu)}
\]

(12.10)

Thus, for a linear isotropic material, only two elastic moduli are necessary to the
elastic deformation. The constitutive equations for linear elasticity are usually writ-
ten using *Lamé’s constants*:

\[
\begin{align*}
\varepsilon_{ij} &= \frac{(1+\nu)}{E} \sigma_{ij} - \frac{\nu}{E} \delta_{ij}\sigma_{kk} \\
\sigma_{ij} &= 2G\varepsilon_{ij} + \lambda\delta_{ij}I_e
\end{align*}
\]

(12.11)

Where \(I_e\) is the first invariant of the infinitesimal strain tensor and \(\delta_{ij}\) is the Kro-
necker delta. The shear modulus, \(G\), is sometimes written using the Greek letter, \(\mu\). The Lamé constant, \(\lambda\), is related to the other elastic moduli by:


\[ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \lambda = K - \frac{2}{3}G \]  \hspace{1cm} (12.12)

**Viscosity**

Over long time spans, even seemingly solid materials creep viscously. In the simplest form of viscosity, a *Newtonian fluid*, shear stress is linearly related to the deformation rate (Fig. 9.3) via the viscosity, \( \eta \):

\[ \tau = \eta \dot{\varepsilon} \]  \hspace{1cm} (12.13)

In a manner similar to elasticity, the constitutive equation for linear viscosity can be written:

\[ \sigma_y = -p \delta_y + \lambda \delta_y \frac{\partial v_k}{\partial x_k} + \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]  \hspace{1cm} (12.14)

Where \( p \) is the *thermostatic pressure*, \( \lambda \) is the *second coefficient of viscosity* and \( \mathbf{v} \) is the velocity. You may recall that the mean pressure is equal to the first invariant of the stress tensor divided by three (Chapter 5):

\[ \sigma_{\text{mean}} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \]  \hspace{1cm} (12.15)

The thermostatic pressure can be related to the mean stress by:

\[ \sigma_{\text{mean}} = -p + \left( \lambda + \frac{2\eta}{3} \right) \frac{\partial v_k}{\partial x_k} \]  \hspace{1cm} (12.16)

**Plasticity**

Unlike the simple, linear forms of elasticity and viscosity theory, plasticity is inherently nonlinear and requires the use of hyperbolic partial differential equations. So, we will not pursue plasticity any more here except to point you towards the classic reference in the field: Hill (1950).
Boundary Conditions and Initial Values

The general procedure for carrying out a mechanical analysis is to solve a set of differential equations that result from manipulation of the physical principles and appropriate constitutive equations by integration. This results in constants of integration that must be evaluated. We have seen an example of this already (although not in the context of a full mechanical analysis): In Chapter 10, the velocity field for the trishear model was derived from the condition of incompressibility and the and an arbitrary choice for the \( v_x \) component of the velocity field. Integrating to solve for the \( v_y \) component resulted in a constant of integration, which we evaluated based on the boundary conditions on the two borders of the trishear zone (Eqn. 10.8).

In general, to solve the integrated differential equations, you must specify either the boundary conditions or the initial conditions. Boundary conditions are limiting values or conditions on the dependent variables at the edges of your model. If you are analyzing the flow of material in a channel, a boundary condition might be that the velocity of the flow must go to zero at the edge of the channel, or in the case of the trishear model just discussed, that the velocity in the triangular shear zone must go to zero on the footwall boundary of the zone. In many problems, one might assume that an important boundary condition is that the surface of the Earth is a traction free surface and thus must be perpendicular to a principal stress. Initial conditions are the values of the time dependent variables at time zero of your analysis. In the case of the flow in the channel, you might specify the velocity of the fluid entering the channel.

Commonly, one specifies either the boundary conditions and solves for that dependent variable in the interior of the body or the initial conditions and solves for the values of that dependent variable at some later time. Imagine that you are studying the formation of a laccolith (Johnson, 1970; Pollard and Fletcher, 2005). You would specify where vertical displacements go to zero (boundary conditions) and, via elasticity theory, solve for the displacement of the bending layer. A problem where the boundary conditions are set is known as a boundary value problem whereas in analyses where the initial conditions are set it is known as an initial value problem. Needless to say, which type of analysis you do is dependent
on what you know already and what your objectives are in the analysis. Take the classic physics problem of a projectile (Middleton and Wilcock, 1994): if you know the mass, angle, and the velocity of the projectile, you know the initial conditions and can calculate how far the projectile should travel and where it will land. From a more practical standpoint, however, you know where the target is that you want to hit (a boundary condition) and you want to calculate the initial velocity and angle that is necessary to hit the target.

Some Simple Geological Examples

There is a rich geological literature of mechanical analysis of structures at various scales and complexity. In this section, however, we will limit ourselves to some simple, yet powerful results, first involving rigid bodies and then from linear elastic fracture mechanics. These results are germane to topics that we have already discussed: thrust belts, hydraulic fracturing and flexure.

Mechanics of Thrust Belts

Hubbert and Rubey’s (1959) Force Balance for Thrust Plates

One of the most famous papers in structural geology was entitled “Role of fluid pressure in mechanics of overthrust faulting” (Hubbert and Rubey, 1959). This paper build on earlier work by M. K. Hubbert (1951) and analyzed the case of a block of geological dimensions that was pushed over a pre-existing surface. Their analysis is a particularly nice example of static equilibrium balance of forces. In their paper, they only balance forces in the $X_1$ direction although Hubbert was clearly aware that this did not constitute a total force balance (Fig. 12.2). Consult Pollard and Fletcher (2005, p. 255-260) if you wish to see the entire three dimensional force balance. The free body diagram (Fig. 12.2) is posed in terms of tractions (stress vectors); we will have to covert these to forces by integrating the traction along the area of interest (i.e., the side or bottom of the block) in order to do the force balance. In the one-dimensional force balance, the normal force on the left side of the block should be equal to the frictional shear force on the base of the block:
Evaluating the right hand side of the equation, recall that the frictional resistance is a function of the normal stress, \( \sigma_{33} = \rho g z \), times the coefficient of static friction (from Byerlee’s law):

\[
\int_0^z \sigma_{11} \, dx_3 = \int_0^x \sigma_{31} \, dx_1
\]  

(12.17)

Evaluating the right hand side of the equation, recall that the frictional resistance is a function of the normal stress, \( \sigma_{33} = \rho g z \), times the coefficient of static friction (from Byerlee’s law):

\[
\int_0^z \sigma_{31} \, dx_1 = \int_0^x \mu \sigma_{33} \, dx_1 = \int_0^x \mu \rho g z \, dx_1 = \mu \rho g z x
\]  

(12.18)

Evaluating the left side of Equation (12.17) requires a slight explanation: There are two possible outcomes as the value of \( \sigma_{11} \) is increased: (a) the push will exceed the frictional resistance to sliding that we have just calculated and the entire block will slide coherently, or (b) \( \sigma_{11} \) will exceed the fracture strength of the material and the block will break up rather than slide as a rigid block. The problem we are going to solve is actually the latter. To do so, we need an expression for the Coulomb Failure criteria in terms of the principal stresses (Eqn. 6.8, repeated here):
\[ \sigma_1 = C_o + K \sigma_3 \quad \text{where} \quad K = \frac{1 + \sin \phi}{1 - \sin \phi} \quad \text{and} \quad C_o = 2S_o \sqrt{K} \] (12.19)

where \( S_o \) is the cohesion and \( \phi \) is the angle of internal friction. Under these conditions, \( \sigma_{11} = \sigma_1 \) and \( \sigma_{33} = \sigma_3 \). Now, we can expand the left side of Equation (12.17):

\[
\int_0^x \sigma_{11} \, dx_3 = \int_0^x (C_o + K \sigma_3) \, dx_3 = \int_0^x (C_o + K \rho g z) \, dx_3 = C_o z + \frac{K \rho g z^2}{2} \] (12.20)

With both sides of Equation (12.17) evaluated, we have:

\[ C_o z + \frac{K \rho g z^2}{2} = \mu_s \rho g x \] (12.21)

Rearranging the result of Equation (12.21) we get an expression for the length of the block, \( x \), in terms of its thickness, \( z \), the friction along the base, etc.:

\[ x_{\text{max}} = \frac{C_o}{\mu_s \rho g} + \frac{K z}{2 \mu_s} \] (12.22)

For \( z = 5 \text{ km} \), \( \rho = 2750 \text{ kg m}^{-3} \), \( \phi = 30^\circ \), and \( \mu_s = 0.85 \), we calculate the maximum length of the block is 11.8 km, which is much less than the dimension of large thrust sheets, such as the Lewis overthrust in Glacier National Park that can be tracked 80 km or more down dip and hundreds of kilometers along strike.

The basic problem is that, at these dimensions, rocks are fundamentally weak as Hubbert demonstrated in an earlier paper when he posed the thought experiment of whether a crane large enough could lift the entire state of Texas! This simple analysis captures the so-called paradox of low-angle thrust faults that structural geologists have been debating since the early 1900’s. Hubbert and Rubey went on to propose that pore fluid pressures, combined with a thrust decollement that dipped gently towards the foreland, could explain large thrust plates. Their work on pore fluid pressures was pioneering but, alas, they were wrong about the dip of the decollement as well as the shape of the thrust block. Our modern understanding of the mechanics of thrust belts as critically tapered wedges is summarized by Dahlen (1990).
Critically-Tapered Wedges (Dahlen, 1990)

In a remarkable series of papers, beginning in 1983 (Davis et al., 1983) and culminating with Dahlen (1990), the Princeton group laid out the modern mechanical basis for understanding thrust belts. Their analysis built on earlier work by Elliot (1976) and Chapple (1978), both of whom recognized that thrust belts in cross section had the form of a finely tapered wedge rather than an rectangular block. In this section, we summarize Dahlen's (1990) general two dimensional force balance in a non-cohesive wedge.

The equations of static equilibrium (force balances) in terms of partial differential equations take into account the z as well as the x direction (Fig. 12.3).

Summing in the x direction first, we get:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} - \rho g z \sin \alpha = 0$$

(12.23a)

and in the z direction:

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g z \cos \alpha = 0$$

(12.23b)
At the upper surface of the wedge, the boundary conditions are: \( z = 0; \sigma_{xz} = 0 \) (i.e., no shear stress on the surface of the wedge); and \( \sigma_{zz} = -\rho g D \) (the weight of the overlying water, or 0 in the case of subaerial wedges). The Hubbert and Rubey pore fluid pressure ratio in the interior of the wedge is given by:

\[
\lambda = \frac{p_f - \rho_f g D}{-\sigma_{zz} - \rho_f g D}
\]  

(12.24a)

and along the base by

\[
\lambda_b = \frac{p_{basal} - \rho_f g D}{-\sigma_{zz} - \rho_f g D}
\]  

(12.24b)

Assuming constant, \( \lambda, \rho, \) porosity, and coefficient of internal friction (\( \mu \)), the components of the stress tensor at any point within the wedge are:

\[
\begin{align*}
\sigma_{xz} &= (\rho - \rho_f) g z \sin \alpha \\
\sigma_{zz} &= -\rho_f g D - \rho g z \cos \alpha \\
\sigma_{xx} &= -\rho_f g D - \rho g z \cos \alpha \left[ \csc \phi \sec 2\psi_0 - \frac{2\lambda + 1}{\csc \phi \sec 2\psi_0 - 1} \right]
\end{align*}
\]  

(12.25)

The angles that the principal stresses make with the upper and lower surfaces of the wedge are:

\[
\begin{align*}
\psi_0 &= 0.5 \left[ \sin^{-1} \left( \frac{\sin \alpha'}{\sin \phi} \right) - \alpha' \right] \\
\psi_b &= 0.5 \left[ \sin^{-1} \left( \frac{\sin \phi_b'}{\sin \phi} \right) - \phi_b' \right]
\end{align*}
\]  

(12.26a)

The primed \( \alpha \) and \( \phi \) are the surface slope and the basal friction angle, modified by the influence of pore fluid pressure.
Out of all this comes an exact and stunningly simple relationship for the critical taper of the wedge (Dahlen, 1990):

\[ \alpha' = \tan^{-1}\left(\frac{1 - \frac{\rho_s}{\rho}}{1 - \lambda}\right) \tan \alpha \]  \hspace{1cm} (12.26b)

\[ \phi_b' = \tan^{-1}\left(\mu_b \frac{1 - \lambda_b}{1 - \lambda}\right) \]

The taper depend on no length parameters and is therefore self-similar. The angles of the principal stresses, and thus the taper, are dependent only on the material properties and the pore fluid pressure and thus do not vary throughout the wedge. Another assumption that we have made is that the entire wedge is on the verge of failure everywhere.

By making a number of small angle assumptions — that \( \alpha, \beta, \psi_0, \) and \( \psi_b \) are all assumed to be much less than 1 — we can recover the initial result of Davis et al. (1983). For subaerial wedges, the approximate expression for the critical taper is:

\[ \alpha + \beta = \psi_b - \psi_o \]  \hspace{1cm} (12.27)

\[ \alpha + \beta = \frac{\beta + \mu_b(1 - \lambda_b)}{2(1 - \lambda)\sin\phi \left(\frac{1}{1 - \sin\phi}\right)} \]  \hspace{1cm} (12.28)

**Step-up Angle of Thrusts**

Now that we know the angles that the principal stresses make with the basal decollement, it is a simple matter to calculate the angle that faults within the wedge will make with respect to the decollement (Dahlen, 1990). From simple Mohr-Coulomb theory, the poles to newly formed faults should form at \( (45 + \phi/2)^\circ \) with respect to the \( \sigma_I \) principal stress direction. Given the above calculations, there are two possible orientations faults (here given as the angle between the fault plane and the basal decollement):
This gives rise to lower angle synthetic thrust faults (forward verging thrusts) and higher angle antithetic thrust faults, i.e., steep back-thrusts (Fig. 12.4). The failure stress on the thrusts within the wedge is:

\[
\delta_b = \left(\frac{90 - \phi}{2}\right) - \psi_b \quad \text{and} \quad \delta_b' = \left(\frac{90 - \phi}{2}\right) + \psi_b
\]  

(12.28)

and on the decollement

\[
\tau_b = \left(\rho - \rho_f\right)gz\sin\alpha\left(\frac{\cos\phi}{\sin 2\psi_0}\right)
\]  

(12.29)

\[
\tau_b = \left(\rho - \rho_f\right)gz\sin\alpha\left(\frac{\sin 2\psi_b}{\sin 2\psi_0}\right)
\]  

(12.30)

The ratio of these two stresses:

\[
0 \leq \frac{\tau_b}{\tau} \leq \frac{\sin 2\psi_b}{\cos \phi} \leq 1
\]  

(12.31)

In other words, the decollement must always be weaker than the wedge. This must obviously be the case or the thrust belt would not move but would break up internally. In fact, if you watch a pile of snow or sand in front of a plow blade, you can observe an alternation between failure of the wedge and sliding on the base.
**Holes and Cracks: Some Important Results from Linear Elastic Fracture Mechanics**

Linear elastic fracture mechanics has provided some deep insights into the deformation in the upper part of the Earth’s crust. The derivations of some of the fundamental equations involve imaginary numbers, *complex variable theory*, and the *Cauchy-Riemann equations*. These are beyond the scope of this manual but the interested student may check out the development in Jaeger and Cook (1976) or McGinty (2015). As you will see, the results of this section are especially germane to subsurface exploration and drilling for hydrocarbons, geothermal, or fluid injection.

**Circular Holes**

The problem of the stresses around a circular hole in a material has been called the “most important single problem in rock mechanics” (Jaeger and Cook, 1976, p. 249). Given that we drill circular holes in rocks for a variety of reasons, it is not hard to see why this is the case! Kirsch (1898) gave the fundamental solution for

![Figure 12.5 — General two dimensional load on a plate with a circular hole. We use a Cartesian coordinate system with X3 parallel to the axis of the hole and pointing into the page. Stresses around the hole are specified in polar coordinate system with the angle θ measured up from the horizontal (inset view). The hole has a diameter of a and the distance from the center of the hole is specified by r.](image-url)
the case of uniaxial loading but here we will go straight to the general two di-

mensional loading case.

We assume a far field coordinate system parallel and perpendicular to the axis of the hole and the far field stresses are defined in that coordinate system (Fig. 12.5). The stresses around the hole are defined in a polar coordinate system as shown in the inset diagram. The hole has a radius, \( a \), and the stresses are calculated at a distance, \( r \), from the center of the hole. The radial and tangential normal and shear stresses are:

\[
\sigma_r = \frac{(\sigma_{11} + \sigma_{22})}{2} \left[ 1 - \left( \frac{a}{r} \right)^2 \right] + \left[ 1 - 4 \left( \frac{a}{r} \right)^2 + 3 \left( \frac{a}{r} \right)^4 \right] \left( \frac{\sigma_{22} - \sigma_{11}}{2} \right) \cos 2\theta + \sigma_{12} \sin 2\theta \]  \tag{12.32a}

\[
\sigma_{\theta\theta} = \frac{(\sigma_{11} + \sigma_{22})}{2} \left[ 1 + \left( \frac{a}{r} \right)^2 \right] - \left[ 1 + 3 \left( \frac{a}{r} \right)^4 \right] \left( \frac{\sigma_{22} - \sigma_{11}}{2} \right) \cos 2\theta + \sigma_{12} \sin 2\theta \]  \tag{12.32b}

\[
\sigma_{r\theta} = \left[ 1 + 2 \left( \frac{a}{r} \right)^2 - 3 \left( \frac{a}{r} \right)^4 \right] \left( \frac{\sigma_{22} - \sigma_{11}}{2} \right) \sin 2\theta + \sigma_{12} \cos 2\theta \]  \tag{12.32c}

For the special case where \( \sigma_{11} = \sigma_1 \) and \( \sigma_{22} = \sigma_2 \) and there is fluid pressure, \( P_f \) in the hole, the \( \sigma_{12} \) term goes zero and the pertinent equations become:

\[
\sigma_r = \frac{1}{2} (\sigma_1 + \sigma_2) \left[ 1 - \left( \frac{a}{r} \right)^2 \right] + \frac{1}{2} (\sigma_2 - \sigma_1) \left[ 1 - 4 \left( \frac{a}{r} \right)^2 + 3 \left( \frac{a}{r} \right)^4 \right] \cos 2\theta + P_f \left( \frac{a}{r} \right)^2 \]  \tag{12.33a}

\[
\sigma_{\theta\theta} = \frac{1}{2} (\sigma_1 + \sigma_2) \left[ 1 + \left( \frac{a}{r} \right)^2 \right] - \frac{1}{2} (\sigma_2 - \sigma_1) \left[ 1 + 3 \left( \frac{a}{r} \right)^4 \right] \cos 2\theta - P_f \left( \frac{a}{r} \right)^2 \]  \tag{12.33b}

\( \sigma_{\theta\theta} \) is commonly referred to as a **hoop stress**.

Figure 12.6 shows how the stresses vary around the hole for the case of uniaxial loading with \( \sigma_1 = 50 \text{ MPa} \) and \( \sigma_2 = 0 \text{ MPa} \). You can see that the tangential or hoop stress at 0° and 180° is three times greater than the far field \( \sigma_1 \) value! Furthermore, at 90° and 270° the stress is tensional. This value of 3 (in uniaxial load conditions) is known as the **stress concentration factor**. Interestingly enough,
this factor is independent of the size of the hole: a small hole produces just as much stress concentration as a large hole. You can see that the stress concentration is very localized near the hole (Fig. 12.6, right). Within a distance of five borehole radii, the hoop stress has dropped to within 2% of the regional value.

Figure 12.6 — Left: Variation of the hoop stresses ($\sigma_{\theta\theta}$) around the edge of a borehole for the case of uniaxial loading. Right: radial and tangential stress variation with distance from a borehole as a function of the radius of the borehole.

Figure 12.7 — The location of breakouts and potential tension cracks around a vertical borehole. The breakouts should parallel the minimum principal horizontal stress direction.
Now, let’s say you are an oil company systematically drilling holes in a producing area. Every one of those boreholes will have these very large stress concentrations and in some cases the stresses will be high enough to cause the well bore to deform by spalling off of pieces in the areas of high stress concentration: these are known as borehole breakouts (Fig. 12.7). For traditional vertical boreholes, breakouts should form by compressive failure in the direction of the least principal horizontal stress and perpendicular to the maximum horizontal principal stress. This turns out to be one of the best ways to determine the orientations of the stress field in the plane perpendicular to the borehole. Figure 12.8 shows the orientations of $\sigma_1$ in the vicinity of the San Andreas fault in central California.

**Cracks**

After Kirsch’s circular hole solution, the theory evolved to elliptical holes which in the extreme case become cracks. Cracks are important because, not only do they determine the ultimate strength of the material, but they also are the site where subsequent failure via fault tip migration occurs. Some complicated math
ensued with each succeeding step forward, but the results are surprisingly simple and powerful. We skip here this important early development and go straight to Irwin’s (1957) approximate solution for stresses near a crack tip, that is $r \leq a/10$ (Fig. 12.9). He calculated that the stresses are:

$$
\sigma_{11} \approx \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 - \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right]
$$  \hspace{1cm} (12.34a)

$$
\sigma_{22} \approx \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left[ 1 + \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{3\theta}{2} \right) \right]
$$  \hspace{1cm} (12.34b)

$$
\sigma_{12} \approx \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{3\theta}{2} \right)
$$  \hspace{1cm} (12.34c)

The quantity $\sigma_\infty \sqrt{\pi a}$ is known as the stress intensity factor:

$$
K_I = \sigma_\infty \sqrt{\pi a}
$$  \hspace{1cm} (12.35)

The above analysis is appropriate for Mode I (opening) cracks (Fig. 6.1). For a Mode II (sliding) crack, the equations are (Pollard and Fletcher, 2005):
\[
\sigma_{11} = \frac{K_n}{\sqrt{2\pi r}} \left[ -\sin\left(\frac{\theta}{2}\right) \left[ 2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] \right] \quad (12.36a)
\]

\[
\sigma_{22} = \frac{K_n}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \quad (12.36b)
\]

\[
\sigma_{12} = \frac{K_n}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] \quad (12.36c)
\]

And, for a Mode III crack:

\[
\sigma_{13} = \frac{K_{III}}{\sqrt{2\pi r}} \left[ -\sin\left(\frac{\theta}{2}\right) \right] \quad (12.37a)
\]

\[
\sigma_{23} = \frac{K_{III}}{\sqrt{2\pi r}} \left[ \cos\left(\frac{\theta}{2}\right) \right] \quad (12.37b)
\]

Figure 12.10 — Two dimensional maps of the \(\sigma_{12} = \sigma_{21}\) magnitude at the tip of a horizontal crack (white line), located at \((0,0)\). Width of each diagram is \(\pm 0.05a\). Color map is different in both but the contour interval is the same. In the Mode I case, blues and greens are opposite signs.
In all of these cases, we find the square root of distance from the crack tip, \( r \), in the denominator. Thus, at the crack tip itself, these equations all suggest infinite stress. Of course, there can’t really be infinite stress there, it is just that linear elasticity doesn’t apply near \( r = 0 \). Figure 12.10 shows plots of the magnitude of shear stress on planes perpendicular or parallel to the crack.

**Final Thoughts: Simulation vs. Illumination**

There are many types of models constructed for different purposes. The fault-related folding models that we saw in Chapter 10, for example, have the goal of simulating the geometry, and occasionally the sequence, of deformation. This is a perfectly reasonable objective and there are many practical reasons why we want to project the geometry to depth using predominantly kinematic rules: to define the geometry of a structural trap in a hydrocarbon reservoir, calculate the amount and distribution of shortening for a palinspastic restoration, evaluate the goodness of fit, or define the likely fault geometry to assess seismic hazard. The goal in simulation is to reproduce, as faithfully as possible, those parts of the structure that we cannot see and what they might look like. Some simulations (e.g., the trishear model) can be carried out extremely rapidly, allowing us to test many possible geometries and find a “best fit” to the data. The problem comes, however, when we assume that the kinematic model “explains” the structure, because we have not, in fact, tested whether the model conforms to the well known physical principles described in this chapter, nor whether the boundary or initial conditions are reasonable.

In this Chapter, we have gotten a glimpse of the suite of physical principles, constitutive equations, and boundary conditions that can be used to illuminate a structural problem of interest. Full mechanical models commonly do not have, as an objective, the simulation of an overall structural geometry. We are not trying to draw a more accurate cross section; instead we are trying to understand why something formed the way that it did. To answer that question, we don't need to reproduce all aspects of the geometry of a structure. What we try to do is distill the problem until all that remains are its most fundamental elements. By making a model simpler, we are more likely to be able to isolate, and illuminate, the key features. When we make models more complex, the number of free variables increases...
to a point that we can no longer say what is most likely or important. Powerful computer packages to carry out large scale numerical models — especially finite element and discrete element models — run the risk of being so complicated that one can no longer isolate, and get insight into, the key parameters.

So, both simulation and illumination have their place in structural geology. The student’s goal is to learn the wisdom to decide what type of model is likely to answer the question of interest.
1. A simple, well known experiment in structural geology that relates directly to the Hubbert and Rubey analysis was proposed by the great French fluid mechanics expert, M. A. Biot, and was called the “beer can” experiment.\(^4\)

(a) Experimental procedure

(i) Take the glass plate provided and place the empty can on one end. Tilt the plate by raising the end on which the can rests. Record the angle of the plate at which the can begins to slide down the tilted glass.

(ii) Cover the plate with a film of water (if the plate is dirty, you will have to clean it with mild detergent first). Repeat step one. Again, record the angle at which the can slides down the plate.

(iii) Now chill the can by placing it in the cooler with dry ice [Safety note: Do not handle the dry ice with your bare hands or you run a serious risk of rapid frostbite. Use gloves to place or remove the can from the cooler. Dry ice — or frozen carbon dioxide — is much colder than ice made from water!].

(iv) After two to three minutes, remove the can from the cooler and place the can on the wetted glass plate with the open end facing up. Tilt until the can slides and record the angle as before.

(v) Finally, cool the can again, briefly, and place it on the wetted plate, open end down. Tilt until the can slides and record the angle.

(b) Draw a two dimensional free body diagram for the beer can experiment.

(c) Use a force balance to calculate the coefficient of static friction between the can and the glass for steps (i) to (v) in part (a). How do you explain the result

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\(^4\) Editorial note: Biot proposed this experiment back in the 1950's, when beer cans were sturdy affairs. Today's beer cans, while being ecologically much more acceptable (though much less welcome on college campuses), have wimpy thin sides that make them unusable for this experiment, so we are reduced to the ignominious fate of having to use a different type of can!
in step (v)? Derive the appropriate equations that demonstrate what is actually going on in step (v) relative to the other steps.

2. The following questions relate to Figure 12.6, which shows the orientations of \( \sigma_i \) in central California.

   (a) Make a sketch showing the average orientation of breakouts in the boreholes that were used to calculate the map of stresses. Be sure to include geographic axes and the orientations of the principal stresses.

   (b) Hydraulic fracturing experiments at a depth of 1.3 km in the region show that \( \sigma_1 \) is 49 MPa with an azimuth of 036° and \( \sigma_3 \) = 25 MPa on an azimuth of 126°. Calculate the hoop and radial stresses around the borehole.

   (c) Assume that the breakouts formed by small Coulomb shear fractures that extend 1.2 borehole radii into the rock (measured from the center of the borehole). What are reasonable values of cohesion and internal friction for the rock mass that would explain the formation of the breakouts?

   (d) The breakouts reported are commonly from depths of around 3700 m. Assuming that the principal horizontal stresses remain the same and \( \rho = 2600 \text{ kg m}^{-3} \), calculate the values and orientations of \( \sigma_1, \sigma_2, \) and \( \sigma_3 \). How would these new values change your answer to part (c)?

3. Figure 12.8 shows the magnitudes of stresses \( \sigma_{12} \) (i.e., \( \sigma_{xy} \)) with distance from the tip of a Mode I and Mode II crack. What exactly does this stress mean? Explain how you would go about calculating the maximum shear stress around the crack tip for the two cases.

4. The following questions apply to the critically tapered wedge theory described earlier in the Chapter.

   (a) Using Equations 12.26 and 12.27, describe what happens as either the static friction on the base of the wedge, \( \mu_b \rightarrow 0 \) or the pore fluid pressure ratio on the base, \( \lambda_b \rightarrow 1 \). The former applies where thrust belt decollements are located in salt horizons; the latter is commonly observed in submarine accretionary prisms.

   (b) The figure on the following page shows two cross sections and topographic profiles across the Subandean fold and thrust belt in Bolivia. The rainfall in the region of the northern profile is 1600 to 2400 mm/yr whereas along the southern profile it is 600-800 mm/yr. Discuss the contrasts between these
two sections/profiles in the context of the critically tapered wedge theory and, where the possible explanations are non-unique, describe what type of data you would like to collect to resolve any ambiguities (assume that money is no obstacle!). You may wish to read Dahlen (1990) first.