Geostrophic Currents

**Full Equations of Motion**

\[
\begin{align*}
\frac{\partial}{\partial t} (u) + u_k \frac{\partial}{\partial x_k} (u) &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + A_H \left( \frac{\partial^2}{\partial x^2} (u) + \frac{\partial^2}{\partial y^2} (u) \right) + A_V \left( \frac{\partial^2}{\partial z^2} (u) \right) + fv \\
\frac{\partial}{\partial t} (v) + u_k \frac{\partial}{\partial x_k} (v) &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + A_H \left( \frac{\partial^2}{\partial x^2} (v) + \frac{\partial^2}{\partial y^2} (v) \right) + A_V \left( \frac{\partial^2}{\partial z^2} (v) \right) - fu \\
\frac{\partial}{\partial t} (w) + u_k \frac{\partial}{\partial x_k} (w) &= -\frac{1}{\rho} \frac{\partial P}{\partial z} + A_H \left( \frac{\partial^2}{\partial x^2} (w) + \frac{\partial^2}{\partial y^2} (w) \right) + A_V \left( \frac{\partial^2}{\partial z^2} (w) \right) - g \\
\end{align*}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{continuity equation resulting from assumption of incompressibility.}
\]
When can acceleration be ignored?

\[
\frac{Du}{Dt} = fv \quad \frac{Dv}{Dt} = -fu
\]

This describes motion that is always being accelerated to the center.

The force balance is between the centripetal force and Coriolis force

\[
\frac{v^2}{r} = \bar{v}f
\]

Where...

\[
\bar{v}^2 = \sqrt{u^2 + v^2}
\]

\[
\frac{v^2}{r} = R_0
\]

\[R_0 = \text{Rossby Number}\]

the ratio of centripetal to Coriolis acceleration

When \(R_0\) is small, acceleration can be ignored and the force balance is between pressure gradient and Coriolis.

Scaling the Equations: The Geostrophic Approximation

We wish to simplify the equations of motion to obtain solutions that describe the deep-sea conditions well away from coasts and below the Ekman boundary layer at the surface. To begin, let’s examine the typical size of each term in the equations in the expectation that some will be so small that they can be dropped without changing the dominant characteristics of the solutions. For interior, deep-sea conditions, typical values for distance \(L\), horizontal velocity \(U\), depth \(H\), Coriolis parameter \(f\), gravity \(g\), and density \(\rho\) are:

\[
L \approx 10^6 \text{ m} \quad H_1 \approx 10^3 \text{ m} \quad f \approx 10^{-4} \text{ s}^{-1} \quad \rho \approx 10^3 \text{ kg/m}^3
\]

\[
U \approx 10^{-1} \text{ m/s} \quad H_2 \approx 1 \text{ m} \quad \rho \approx 10^3 \text{ kg/m}^3 \quad g \approx 10 \text{ m/s}^2
\]

where \(H_1\) and \(H_2\) are typical depths for pressure in the vertical and horizontal. From these variables we can calculate typical values for vertical velocity \(W\), pressure \(P\), and time \(T\):

\[
\frac{\partial W}{\partial z} = - \left( \frac{\partial U}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

\[
\frac{W}{H_1} = \frac{U}{L} ; \quad \frac{W}{H_1} = \frac{UH_1}{L} = \frac{10^{-1} \times 10^3}{10^6} = 10^{-4} \text{ m/s}
\]

\[
P = \rho g H_1 = 10^3 \times 10^1 \times 10^3 = 10^7 \text{ Pa} ; \quad \frac{\partial P}{\partial x} = \rho g H_2 / L = 10^{-2} \text{ Pa/m}
\]

\[
T = L / U = 10^7 \text{ s}
\]
Momentum Equation for Vertical Motion

The momentum equation for vertical velocity is therefore:

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \varphi - g
\]

\[
\frac{W}{T} + \frac{U W}{L} + \frac{U W}{L} + \frac{W^2}{H} = \frac{P}{\rho H} + f U - g
\]

\[10^{-11} + 10^{-11} + 10^{-11} + 10^{-11} = 10 + 10^{-5} - 10\]

and the only important balance in the vertical is hydrostatic:

\[
\frac{\partial p}{\partial z} = -\rho g \quad \text{Correct to } 1:10^6.
\]

This is also known as the hydrostatic equation.

Momentum Equation for Horizontal Motion

The momentum equation for horizontal velocity in the x direction is:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + f v
\]

\[10^{-8} + 10^{-8} + 10^{-8} + 10^{-8} = 10^{-5} + 10^{-5}\]

Thus the Coriolis force balances the pressure gradient within one part per thousand. This is called the geostrophic balance, and the geostrophic equations are:

\[
\frac{1}{\rho} \frac{\partial p}{\partial x} = f v; \quad \frac{1}{\rho} \frac{\partial p}{\partial y} = -f u; \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = -g
\]

This balance applies to oceanic flows with horizontal dimensions larger than roughly 50 km and times greater than a few days.

The equations can be written:

\[
u = -\frac{1}{f \rho} \frac{\partial p}{\partial y}; \quad v = \frac{1}{f \rho} \frac{\partial p}{\partial x}
\]
Geostrophic Equations

\[ \frac{1}{\rho} \frac{\partial p}{\partial y} = -fu \]
\[ \frac{1}{\rho} \frac{\partial p}{\partial x} = f v \]

Expresses the balance between Pressure Gradient and Coriolis force - This is often referred to as Geostrophic Balance.

\[ u = -\frac{1}{\frac{f}{p}} \frac{\partial p}{\partial y} \quad \quad v = \frac{1}{\frac{f}{p}} \frac{\partial p}{\partial x} \]

These expressions provide a means for determining current velocity from the pressure field which is related to the distribution of density in the ocean. These currents are referred to as Geostrophic Currents.

Note: Wherever you find strong pressure gradients in the ocean you will have fast moving geostrophic currents.

Pressure Field in the Ocean

Two ways to express pressure field in the ocean: 1) give the pressure on a series of level surfaces (with respect to the earth's geopotential surfaces) or 2) give the geopotential heights of equal pressure surfaces.

(For more information see Stewart 153...)

\[ \frac{dp}{dz} = \rho g \]
\[ \frac{1}{\rho} \frac{dp}{dz} = g dz = dD \]
\[ \frac{1}{\rho} \frac{dp}{dx} = \frac{dD}{dx} \]

D is the Geopotential Height

Then...

\[ u = -\frac{1}{f} \frac{\partial D}{\partial y} \quad \quad v = \frac{1}{f} \frac{\partial D}{\partial x} \]
Geopotential height between two equal pressure surfaces \( p_1 \) and \( p_2 \):

\[
\frac{p_2}{p_1} \frac{1}{\rho} dp = \int_{p_1}^{p_2} dD = D
\]

Geopotential height between two equal pressure surfaces \( p_1 \) and \( p_2 \) at two different hydrographic stations \((a \text{ and } b)\), \( \alpha = 1/\rho = \) specific volume:

\[
\frac{p_2}{p_1} \alpha_a dp = D_a \quad \frac{p_2}{p_1} \alpha_b dp = D_b
\]

Difference in geopotential height between two equal pressure surfaces \( p_1 \) and \( p_2 \) at two different hydrographic stations:

\[
\frac{p_2}{p_1} \int (\alpha_a - \alpha_b) dp = D_a - D_b
\]

Slope between two surfaces of constant pressure:

\[
\frac{D_a - D_b}{\Delta X}
\]

Geostrophic at pressure surface \( a \) relative to the geostrophic velocity at pressure level \( b \). Note that you need a third hydrographic station to get the slope of \( D \) in the north/south direction to compute the \( u \)-component of the geostrophic velocity shear.

\[
v = \frac{1}{f} \frac{D_a - D_b}{\Delta X}
\]
Figure 10.9. Mean geopotential anomaly relative to the 1,000 dbar surface in the Pacific based on 34,356 observations. Height of the anomaly is in geopotential centimeters. If the velocity at 1,000 dbar were zero, the map would be the surface topography of the Pacific. After Wyrski (1979).
Thermocline Motion in Response to Surface Water Convergence and Divergence
Stewart Gives these important steps...

1. Calculate the $D_a - D_b$ between P1 and P2.

2. Calculate the slope of the upper pressure surface relative to the lower pressure surface.

3. Calculate the geostrophic current at the upper surface relative to the lower surface. This is the current shear.

4. Integrate the current shear to a depth of known current speed
   a) surface current (derive from altimetry) downward
   b) depth of no motion upwards
atmospheric density is small

\[ \rho_1 f v_1 = \frac{\partial \rho}{\partial x} \frac{(\rho_1 - \rho_0) g \nabla Z_1}{\nabla X} = \rho_1 g i_1 \]

\[ \rho_2 f v_2 = \frac{\partial \rho}{\partial x} \frac{(\rho_1 - \rho_0) g \nabla Z_1 + (\rho_2 - \rho_1) g \nabla Z_2}{\nabla X} \]

\[ i_2 = \frac{f}{g} \left( \frac{\rho_2 v_2 - \rho_1 v_1}{\rho_2 - \rho_1} \right) \quad i_n = \frac{f}{g} \left( \frac{\rho_n v_n - \rho_{n-1} v_{n-1}}{\rho_n - \rho_{n-1}} \right) \]

\[ i_n = -\frac{f}{g} \left( \frac{\rho_{n-1}}{\rho_n - \rho_{n-1}} \right) \quad i_n = -i_{n-1} \left( \frac{\rho_{n-1}}{\rho_n - \rho_{n-1}} \right) \]

Special case of bottom layer velocity = 0

Several Different Variations on Water Column Density Structure given by Knauss...

\( \rho_1 \times \)
\( \rho_2 \bigcirc \)
\( \rho_3 \times \)

\( \rho_1 \times \)
\( \rho_2 \bigcirc \)
\( \rho_3 \times \)

\( i \) slope of the isopycnal surface  \( \times \) flow into page  \( \bigcirc \) No flow