Ekman Layer Processes

Equations of Motion

Under steady state conditions it can be shown that in the boundary layer of the upper ocean (order hundred meters) horizontal gradients are small compared to vertical gradients. Under these conditions, there is a balance between Coriolis and Friction.

\[
\frac{\partial u}{\partial t} + u_k \frac{\partial u}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_H \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + A_V \left( \frac{\partial^2 u}{\partial z^2} \right) + f\nu
\]

\[
\frac{\partial v}{\partial t} + u_k \frac{\partial v}{\partial x_k} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_H \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + A_V \left( \frac{\partial^2 v}{\partial z^2} \right) + f\nu
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

continuity equation resulting from assumption of incompressibility.
Solutions Derived from the Boundary Layer
Balance between Friction and Coriolis

\[-fv = Ah \frac{\partial^2 u}{\partial z^2} \quad fu = Ah \frac{\partial^2 v}{\partial z^2}\]

Find solution to these two coupled differential equations to obtain expressions for \(u\) and \(v\) as a function of surface wind stress and latitude (i.e., Coriolis) - **Ekman Spiral**

Integrate \(u\) and \(v\) over depth to find the horizontal mass and volume transport - **Horizontal Ekman Transport**

Use continuity equation to determine **Ekman Pumping/Suction** from divergence in the Horizontal Ekman Transport

\[\text{Note: } u \text{ and } v \text{ are velocity components relative to northward directed surface wind stress } \tau = \tau_{yz}\]

Stewart page: 139-
Solution to Boundary Layer Equations For a Northward-Directed Wind Stress

\[ u = V_0 e^{\frac{\pi z}{4}} \cos \left( \frac{\pi z}{4} + az \right) \]
\[ v = V_0 e^{\frac{\pi z}{4}} \sin \left( \frac{\pi z}{4} + az \right) \]

for \( z = 0 \)...

More generally for various depths...

Development of Force Balance Between Friction and Coriolis in the Ocean’s Boundary Layer
**Ekman Depth**

Ekman depth is *functionally defined* as the depth at which the current moves in the opposite direction of the wind stress.

\[ u = V_0 e^{\alpha z} \cos \left( \frac{\pi}{4} + az \right) \]

\[ v = V_0 e^{\alpha z} \sin \left( \frac{\pi}{4} + az \right) \]

**Note:** find \( z (D_E) \) such that cosine terms = -1 and sine term is 0.

\[ D_E = \frac{\sqrt{\frac{2\pi A_z}{f}}}{\cos \phi} \]

\[ V_0 = \frac{\tau}{\sqrt{\rho \nu f A_z}} \]

\[ \tau = \rho_0 u_0 C_D U_{10}^2 \]

**Note:** Ekman Depth is a function of wind speed and latitude.

**Typical Values of Ekman Depth**

<table>
<thead>
<tr>
<th>Wind Speed (( U_{10} )) (m s(^{-1} ))</th>
<th>Latitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15(^0)</td>
</tr>
<tr>
<td>5</td>
<td>75 m</td>
</tr>
<tr>
<td>10</td>
<td>150 m</td>
</tr>
<tr>
<td>20</td>
<td>300 m</td>
</tr>
</tbody>
</table>
**Integrated Mass Transport in the Ekman Layer**

\[ A_h \frac{\partial^2 u}{\partial z^2} = -fv \]
\[ \frac{1}{\rho} \frac{\partial \tau_x}{\partial z} = -fv \]
\[ \int_0^z \frac{\partial \tau_x}{\partial z} dz = -f \int_0^z \rho v dz \]
\[ \tau_w^x = -fM_y \]

\[ A_h \frac{\partial^2 v}{\partial z^2} = fu \]
\[ \frac{1}{\rho} \frac{\partial \tau_y}{\partial z} = fu \]
\[ \int_0^z \frac{\partial \tau_y}{\partial z} dz = f \int_0^z \rho u dz \]
\[ \tau_w^y = fM_x \]

**definition of terms:**
- Surface wind stress in x and y direction
- \( \tau_w^x, \tau_w^y \)

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**Ekman Pumping/Suction**

Take derivatives with respect to y and x, respectively...

\[ \frac{1}{\rho} \frac{\partial \tau_x}{\partial z} = -fv \quad \rightarrow \quad \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\partial \tau_x}{\partial y} \right) = -\frac{\partial}{\partial y} (fv) = -f \frac{\partial v}{\partial y} - v \frac{\partial f}{\partial y} \quad \text{Eq 1} \]

\[ \frac{1}{\rho} \frac{\partial \tau_y}{\partial z} = fu \quad \rightarrow \quad \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} \right) = \frac{\partial}{\partial x} (fu) = f \frac{\partial u}{\partial x} \quad \text{Eq 2} \]

Subtract Eq 1 from Eq 2...

\[ \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) = f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \nu \frac{\partial f}{\partial y} \]

\[ \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) = f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + v \beta \quad \text{where...} \quad \beta = \frac{1}{R_E} 2\Omega \cos(\phi) \]
Ekman Layer Response to Surface Wind Forcing

Given a measure surface wind stress you can compute the horizontal components of the total mass transport integrated over the the Ekman Layer.

\[ M_x = \frac{1}{f} \tau_w^y \quad M_y = -\frac{1}{f} \tau_w^x \]

Given a measure surface wind stress you can compute the vertical Velocity at the Base of the Ekman Layer. Downward Velocity = Ekman Pumping, Upward Velocity = Ekman Suction

\[ \frac{1}{\rho_0 f} \left( \frac{\partial \tau_w^y}{\partial x} - \frac{\partial \tau_w^x}{\partial y} \right) + \frac{\beta}{\rho_0 f^2} \tau_w^x = w_E \]
Ekman Volume Transport

Ekman Volume Transport ($Q$) is just the Ekman Mass Transport ($M$)
divided by density

$$Q_x = \frac{1}{\rho_0} M_x = \frac{1}{\rho_0 f} \tau^y_w$$

$$Q_y = \frac{1}{\rho_0} M_y = -\frac{1}{\rho_0 f} \tau^x_w$$

The volume transport expression can be combined with any
intensive quantity (e.g., surface nitrate concentration) to determine
its horizontal flux from wind stress measurements.

Homework Assignment

Starting with the Expressions for Horizontal Mass Transport...

$$M_x = \frac{1}{f} \tau^y_w$$

$$M_y = -\frac{1}{f} \tau^x_w$$

Show that the the Curl (The Determinant in Linear Algebra
Language) of the Surface Wind Stress will produce a Divergence
in the Horizontal Mass Transport.

hint to get started...

$$M = (m_x \hat{i}, m_y \hat{j}, 0 \hat{k})$$

$$\tau = (\tau_x \hat{i}, \tau_y \hat{j}, 0 \hat{k})$$
**Ekman Transport**

\[
\frac{\tau_y}{\rho f} = \int u_e dz = U_{\text{Transport}}
\]

\[
-\frac{\tau_u}{\rho f} = \int v_e dz = V_{\text{Transport}}
\]

**Wind Stress from QuikScat**
Ekman Transport

![Maps showing Ekman Transport from 2000 to 2003.]

Monthly Nitrate Concentration
(NOAA/NODC-WOA01 Climatology)

![Images of nitrate concentration for each month from January to December.]

Volume Transport (m² s⁻¹)

Nitrate Concentration (µM)
Monthly Ekman Transport

Ekman Transport of Nitrate

\[ \frac{\partial N_{ek}}{\partial t} = \nabla \cdot \left( \bar{U}_{trans} N_{surface} \right) \]