In plate bending problems a flat earth is used as an approximation. Spherical shell theory is more complicated, and is rarely applied to lithospheric bending problems. Bending effects can be considered on two scales. One is a scale of several hundred km or less appropriate for a cross sectional 2-D view of a trench profile, for example. On this scale sphericity is a minor effect. However, for a 3-D view of an entire island arc segment, i.e. over scale lengths of hundreds to several thousand km, sphericity becomes important and is probably essential to understanding why subduction zones are often arcs in plan (map) view.

The mechanical behavior of the lithosphere and asthenosphere is idealized as an elastic plate floating on a fluid. The plate is the lithosphere, not simply the crust, and often includes part of the uppermost mantle. This idealized rheology is appropriate at time scales long enough that the elasticity of the asthenosphere is negligible, i.e., the shear stresses in the asthenosphere are relaxed and a nearly hydrostatic equilibrium prevails. The earth's gravity field induces a hydrostatic pressure field in the fluid. If a weight (mass times $g$) is placed on top of the elastic plate, the plate bends down until the pressure exerted by the fluid on the underside of the plate, integrated over the extent of its deflection, balances the weight of the load. The reason this pressure balances the load is that the fluid is not permitted to leak out of the lithospheric "skin". Thus the downwards deflection produces a net upwards directed force (pressure times area) that supports the load. The role of the elastic plate in this hydrostatic equilibrium is to spread out the bouyant response of the liquid to the load well beyond the boundaries of the load itself.

As usual, the simplest approach is to consider a 2-D structure, i.e. one where nothing changes along strike, so that the cross-section shows everything of interest. We consider two cases. In one, the plate is unbroken and continuous, and in the second, a half plate is considered, with forces applied to its end. What happens to the other side of the medium is not considered in the second model, but it is assumed that the asthenospheric fluid below is not permitted to flood in over the downbent plate. These are two well posed problems in mechanics, and have simple solutions.

The figure below shows the case of a continuous plate. The vertical plot is out of scale to emphasize the main features of the deflection. The horizontal coordinate is horizontal distance, $x$. The deflection of the plate, $\Delta W$, is measured positive downwards and is measured relative to an unloaded and undeflected configuration.

![Diagram showing deflection of an elastic plate](image)

line load = vertical force per unit length
= mass per unit length times $g$

SL

water

fluid asthenosphere

undeflected position

elastic plate

arrows show pressure exerted on plate as a result of its displacement

$H$

$\Delta w$
The pressure is related to the deflection as follows. For a given deflection \( \Delta W \), the unit area column of asthenosphere with height \( \Delta W \) is replaced with a similar sized water column. Since the weight of the water column does not balance the pressure in the asthenosphere beneath the plate, a net force is applied to the unit area of the column at the bottom of the plate. This upwards directed pressure is \( \Delta W (\rho_m - \rho_w) g \). Integrating this pressure over the extent of deflection produces a net force upwards that balances the downwards \( F \).

The bending profile is given as follows:

\[
\Delta W = \frac{F}{\sqrt{2 (\rho_m - \rho_w) g a}} \cdot e^{\frac{a}{4}} \cos \left( \frac{x}{a} - \frac{\pi}{4} \right)
\]

where

\[
a = \frac{4 \sqrt{\frac{4 D}{(\rho_m - \rho_w) g}}}{4}
\]

The load is idealized as a line force \( F \) (force per unit length along strike). This force/length is

\[
\text{force/length} = \text{area} \cdot \text{density} \cdot \text{gravity}
\]

\( a \) is the "flexural parameter", with dimensions of length, which determines the wavelength of the bending. \( a \) is fixed by the elastic properties of the plate through the term \( D \), the "flexural rigidity" of the plate (a measure of its resistance to bending). The effect of the fluid pressure on the underside of the plate is given by the term in the denominator (\( g \) is the gravitational acceleration). \( D \) is given in terms of Young's Modulus, \( E \), the thickness of the plate \( H \), and Poisson's ratio \( \nu \), as follows:

\[
D = \frac{EH^3}{12 (1 - \nu^2)}
\]

A plot of \( \Delta W \) normalized to \( \Delta W_0 \) (the scaling term multiplying the exponential and cosine terms in the equation for \( \Delta W \) above) versus \( x/a \) is given below, where

\[
\Delta W_0 = \frac{F}{(\rho_m - \rho_w) g a}
\]
Also shown is the curve for the half plate with end line load \( F \). This is given by

\[
\Delta W = \Delta W_0 e^{-\frac{x}{a}} \cos \left( \frac{x}{a} \right)
\]

where again the fluid is not allowed to move around the end and fill up the downwards deflected area. This type of model is used for the outer wall of a trench.

In both cases the horizontal extent of the bending is fixed by the flexural parameter, \( a \), which in turn mainly depends upon the thickness, \( H \), of the elastic plate as defined above. A plot of \( a \) versus \( H \) is shown below, with \( E = 10^{12} \) dynes/cm\(^2\) and \( \rho_m = 3.35 \) gm/cm\(^3\), \( \rho_w = 1.0 \) gm/cm\(^3\), and \( Pr = 0.25 \). Note that \( a \) depends upon whether the bending is covered by water or not.

Since the deflection right beneath the load is not large enough to locally compensate the load (the compensation is spread out by the strength of the plate), there is an excess of mass right at the point of application of the load. Note that in the mathematical model the load is idealized as a line load occupying only a point in the cross section. On the flanks of the load, where the plate is deflected downwards, there is a deficiency of mass as noted above. For typical \( a \) values of the order of 100 km, and deflections very much smaller than this, the free-air anomaly is roughly approximated by the effects of the mass deficiencies in the column beneath the observation point as given by the infinite slab formula, \( 2\pi G(\text{density})(\text{height of column}) \) or \(-2\pi G(\rho_m - \rho_w)\Delta W\). Near the
load the excess mass will override this to cause a net positive free air anomaly, the shape of which will depend upon the actual geometry of the load.

\[ a, \text{ the flexural parameter, km} \]

\[ \text{Thickness of elastic plate, km} \]