Understanding Natural Systems through Simple Dynamical Systems Modeling

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Abstract

Undergraduate students at all levels are encouraged to "think critically." We work toward this goal by connecting the theoretical concepts of computer modeling to a simple physical model constructed in lab. Students build a physical representation of a one-box model, consisting of reservoir, input and output. They observe the steady state system, then perturb the system by instantaneously doubling the rate of input. Students observe the response of the system as it arrives at a new steady state with increased input, output and volume. Students then construct a computer model of the same system, and observe that the computer produces results analogous to their observations. In subsequent exercises they modify their models in order to study more complicated systems. There are several advantages to this approach. 1) Students gain confidence in computer modeling by working with a system that they already understand well. 2) Computer modeling is a creative exercise in which students must both analyze a situation and design its solution. 3) The one-box model is a flexible system which is easily adapted for study of a wide variety of natural phenomena. 4) Using appropriate software, the model building can be carried out by introductory-level students, and 5) the mathematical description of a one-box model is well within the grasp of students with basic calculus skills, and thus geology majors can gain experience in applying mathematical techniques to problems of practical interest. This exercise is open-ended--students begin on a firm platform of observation and are armed with theory and computer skills, thus allowing them to explore a range of problems whose solutions require creative and critical thinking.

Keywords: Earth systems science; education--computer assisted; education--undergraduate; geology teaching and curriculum.

Introduction

Dynamical systems models are powerful tools for studying many phenomena in Earth Science. These models (often rather sophisticated) are in increasingly wide use as research tools in hydrology, geochemistry, petrology, oceanography and climatology. This article is not about such models, but rather about simple dynamical models and their use in developing insight into seemingly complex physical phenomena. For both students and researchers, simple systems models can be a tremendous aid for developing insight into the dynamic behavior of natural systems (e.g. deWet, 1994; Kinder & Mosely-Thompson, 1994). However, the mathematical and computer skills necessary to develop dynamical systems models have been beyond the abilities common among undergraduates. Fortunately, recent advances in computer software have dramatically lowered the barriers to systems modeling. Furthermore, computer models developed in combination with simple physical models allow undergraduate students to better understand systems behavior. They gain confidence in computer modeling, and learn how theoretical models can be extended to study more complicated systems. In this way students are provided with a flexible set of tools that can be applied to a wide range of problems, and are nudged toward "critical thinking," the theme that has become so central to undergraduate education.

We present here a combined approach in which identical physical and computer models are constructed by students in successive lab periods. This technique is applicable to undergraduate students at any level; the example given here was developed for an introductory "Earth systems" course populated primarily by non-science majors. Students use a Coleman cooler to construct a physical representation of a one-box model, then design a computer model of the same system with Stella II® software, in order to study steady state and non-steady state behavior. Thus students gain an understanding of an important concept applicable to a wide variety of behaviors in natural systems. In subsequent lab periods students modify and expand their model so that they may study more complicated systems. We describe here both the construction of the physical model and the mathematical basis for the phenomena observed by the students. We then detail the construction of the cooler model using Stella II, and give examples of extensions and other applications of this model. One of the most difficult skills for science students to learn is the ability to go from observation to theory. We believe that the combination of observation, mathematical description and computer modeling of a simple but interesting system allows students to 1) see the utility of modeling natural systems and 2) understand how a meaningful model is constructed. By beginning with a firm tie between their theoretical work and a physical system that they understand, students are better able to make the transition to the more abstract problems and concepts that are so important in all areas of inquiry.

The Cooler Model

The system that we will study is a simple one-box model, consisting of reservoir, input and output. A Coleman cooler partly filled with water represents the reservoir. Opening the drain tap on the cooler provides an output, and running water from a hose provides the input. The first step of the lab exercise is to observe steady state behavior. At "steady state" the volume of water in the cooler is constant, therefore the input must equal the output. We calibrated the volume of the cooler prior to the lab experiment, marking the inside of the cooler in 2 liter increments. The cooler is set up to drain into the sink. Our cooler holds 32 liters of water, and experimentation with our system showed that a good starting point was an initial volume of 8 liters, with equal input and output of 0.011 liter/sec. As our lab time is limited, we have the cooler set up and running as a steady state system when the students enter lab (Figure 1a). The students then observe the steady state system with constant volume, input and output, and calculate the residence time for this system (see below). In order to measure output we use a stopwatch and a one liter beaker, measuring the time required for one liter of water to leave the cooler. The students then calculate the output rate.

The important point for the students to understand about the steady state system is that although water continually enters the reservoir and continually flows out, none of the parameters of the system change with time. This system is dynamically maintained, as opposed to one that is static. The input and output will not change and the volume of the system will remain at 8 liters indefinitely. One question we ask the students is, "How long does it take for water to flow through the system?" The residence time ($\tau$) is given by the volume of water in the reservoir divided by the rate of input. It is an estimate of the average time an atom (or particle etc.) spends in the reservoir between input and output (note that at steady state input = output so that $\tau$ could equivalently be defined as volume/output). However, observation of a system at steady state, no matter how dynamic, does not make for a very exciting lab period.

What if we were to change the parameters of the system? What would happen if we were to double the amount of water input to the reservoir? This is easily achieved by adding a second hose connected to a second faucet running at the same rate as the first. At this juncture we ask the students to make a prediction (in writing) about the behavior of the system. The input is then doubled and the students graph the results. They are asked to graph the volume of the reservoir as a function of time, the time required for a liter of water to leave the cooler, and the corresponding output rate. In our lab the students make measurements at three-minute intervals and observe the system for thirty minutes. Teams of at least three students each are desirable for accurate results.

What is the behavior of this non-steady state system? The most common prediction made by students is that the cooler will overflow. However what they learn (which is in fact something that most of them knew but simply didn't think about) is that the rate of output is dependent on the volume of the cooler. The more water, the faster it flows out. The students observe that the volume of the reservoir and the output rate increase, but that the rate of change declines exponentially with time. The reservoir volume and output rate gradually approach new steady state values, where the new output equals the doubled input (Figure 1b). Students then calculate the new residence time for this system. What they have observed is the response of a system initially at steady state to an input perturbation, resulting in an
exponential approach to a new dynamic equilibrium or steady state.

Mathematical Relations Governing Reservoirs with Feedbacks

There are several approaches to understanding the behavior of a system like the one we have described above. The first is phenomenological—we simply note that as the cooler fills, water flows out of it faster, and at some point this effect balances the input. This is a qualitative description, and is an important step. If we like, we could plot the output data on a graph, and fit a curve through the points. This would allow us to "quantify" our observations—we would now be able to predict the rate of outflow for any amount of water in the cooler. However, we would not have a physically based explanation for why the outflow depends on the volume, nor could we use our data to accurately predict the behavior of some other cooler. And, in order to explore other features of the cooler's behavior, such as how the time to reach steady state depends on the input change, we would be forced to perform a number of time-consuming experiments. Thus, while having generated some ability to predict quantitatively, we would not easily be able to generalize our results to other, similar systems. To do so, we must have a process-based model of the cooler's behavior. In fact, what we would really like to have is some general understanding of how systems like the cooler behave. In this section, we develop a mathematical description of a system that shares some of the features and exhibits some of the behavior of our cooler. In doing so, we can learn a great deal that is generalizable about systems with feedback.

The analytical approach permits a deeper understanding of the behavior of a simple dynamical system. We include this for the benefit of the instructor and for more advanced students. Most students majoring in geology are required to take 1-2 semesters of calculus. While faculty require this level of mathematical training from students, often we do not take advantage of it in a traditional geology curriculum. Here we present a situation where the students can see how mathematical techniques they have studied can be used to develop important understanding of a general class of phenomena with many geological applications. We wish to emphasize that the computer modeling and exercises developed below do not require that the students understand any calculus at all. For introductory students, the following section is unnecessary. However, we feel strongly that more advanced students will derive the greatest benefit and deepest understanding from their modeling effort if it is coupled with a mathematical approach. Conversely, the mathematics are much more likely to be understood if they are used to describe a system that the students understand and are interested in.

Again, the system we consider is that of a single reservoir with an input and an output of some component. We hold the input rate \( a \) constant and define the output rate to be linearly dependent on the quantity \( n \) of said component in the reservoir.

From closely analogous problems in the chemistry of reaction rates, this is known as "first order" kinetics, where the output rate depends on the first power of the reservoir content, \( n \). We can state this mathematically as:

\[
\frac{dn}{dt} = a - kn
\]

where \( k \) is known as the rate constant. Note the similarity of this equation to the familiar equation for radioactive decay:

\[
\frac{dn}{dt} = -\lambda n
\]

The only difference is the addition of the input term, \( a \), and \( k \) takes the place of the familiar radioactive decay constant, \( \lambda \). Eqn. (1a) may be solved as a function of time by separation of variables and integrating:

\[
\int \frac{dn}{a - kn} = \int dt
\]

We can evaluate this integral with initial conditions \( t = 0 \), and \( n = n_o \). It can be seen that the solution (3a) is related to the solution for the radioactive decay equation (3b):

\[
(3a) \quad n(t) = \frac{a}{k} \left[ 1 - e^{-kt} \right] + n_o e^{-kt}
\]

\[
(3b) \quad n(t) = n_o e^{-\lambda t}
\]

The last term in Eqn. (3a) is that of the decay equation, where \( n_o \) is the initial quantity (at time = 0) in the reservoir, directly analogous to the initial quantity of radioactive atoms in a mineral. We can see that \( k \) must have units of \( t^{-1} \), as does \( \lambda \).

It is helpful to rewrite Eqn. (3a) in order to better understand its behavior (4):

\[
(4) \quad n(t) = \frac{a}{k} \left[ \frac{a}{k} - n_o \right] e^{-kt}
\]

Here the first term on the right-hand side, \( a/k \), represents the system at steady state, while the second term describes the non-steady state behavior of the system. As \( t \) becomes large, \( e^{-kt} \) becomes small, so for long times the solution simplifies to:

\[
(5) \quad n(t) = \frac{a}{k}
\]
What does this mean, exactly? In our model \( a \) and \( k \) are constants, so as \( t \) becomes large, the quantity \( n \) in the reservoir also becomes constant. Thus the model tends to "steady state," where the input is exactly balanced by the output, and no further change takes place. This is true despite the fact the system is dynamic--material flows into and out of the reservoir. By manipulating Eqn. (5) we can see that, at steady state:

\[
\frac{1}{k} \equiv \tau = \frac{n}{a}
\]

We note that the residence time, \( \tau \), previously defined as \( n/a \), is also equal to the inverse of the output rate constant \( k \). The analytical approach shows us that the residence time is a very important parameter for a simple system such as ours. Not only does it describe the average transit time of a component through the reservoir, but it also tells us how quickly the system will adjust to a perturbation of the input. The response time, the time necessary for a system such as ours to "substantially adjust to new conditions," may be defined as the time necessary for the exponential term to reach \( e^{-1} \). Examination of Eqn. (4) shows that this will occur when \( t = 1/k = \tau \). Thus, in this simple case the response time and the residence time are equal. This need not be true for more complicated systems, however (see the classic article by Lasaga, 1981, who presents an excellent and in-depth discussion of systems modeling applied to geochemical systems). A second important result of this analytical approach is that we can see the response time of our first order (linear) system to a perturbation does not depend on the size of the perturbation--in other words, a doubling of the input rate to our box model will not cause it to approach steady state any faster than a tripling of the input would. The steady state reservoir volume will, of course, be different.

Clearly, some real systems are characterized by more complicated relationships between input, output and reservoir size. In fact, we will suggest some interesting examples below. However, integrating Eqn. (2) can be difficult or impossible in the case of more complex relationships (Holland (1978) and Lasaga (1981) present analytical solutions for several other cases), and in practice many systems are solved with numerical methods on a computer. The Stella II software obviates the need for the user to understand the numerical solution of equations arising from box modeling. Below, we show how Stella II can be used even by introductory students to investigate the behavior we have described above, and excitingly, to extend their investigations to more complicated systems, all the while focusing their energies on understanding the dynamics of the system and not on solving differential equations.

**Computer Models with Stella II Software**

Hands-on experience in the lab with a one-box model is a very effective way for students to understand the behavior of that particular system, and many students are able to draw appropriate analogies between their experiment and other similar systems. However through the use of computer models constructed with Stella II, students can extend their understanding far beyond the system that they have observed to analogous systems and to more complicated systems. We feel that the best way for students to begin their experience with computer modeling is to construct a model of the same system that they constructed with their physical apparatus.

We give the students an exercise based on the tutorial that comes with Stella II. As the students work their way through the tutorial and become familiar with the software, they also construct a one-box model of their cooler system. Just as they did in the lab, they create a reservoir with an input and an output (Figure 2). Stella II allows the output to be dependent on the volume of the reservoir. In this case we build a model with a linear dependence on volume; output = volume/\( \tau \). The parameters in the Stella model (initial volume, input and \( \tau \)) can be set to the same values that the students obtained from their cooler experiment. Using Stella's graphing tool, students demonstrate to themselves that they have reproduced the initial steady state system.

Next, the Stella model is modified to include a sudden doubling of the input rate using the STEP function. The students again verify that the computer model reproduces the behavior of the analogous cooler experiment (Figure 3). An advantage of computer modeling that becomes immediately apparent is that the Stella simulation is much quicker than the lab experiment. At this point, students can answer questions that they may have had earlier, such as "What happens if the input is tripled rather than doubled?" As stated previously, an important result of this modeling exercise is the observation that although the steady state volume will depend on the size of the perturbation, the time necessary to adjust to a new steady state is the same for any perturbation. With
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**Applications and Extensions**

The system is set to run for "10 minutes" at the original steady state, then input is instantaneously doubled. Parameters input into the model for this perturbation are: Cooler = 8, Input = 0.011+STEP(0.011,600), Output = Cooler/720. Note that both cooler volume and input and output rates are displayed on the y-axis.

Stella II, even students with no math background can demonstrate this for themselves.

Observant students will notice that the reservoir volume calculated by the Stella model is quantitatively different from their experimental result. This is because output from the cooler is not a linear function of water volume. We address this issue in the following section. However, the qualitative behavior is the same, and students can easily see that a very simple computer model has reproduced the most salient features of their laboratory experiment.

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**Applications and Extensions**

The discrepancy between the computer model and its cooler counterpart opens the door to numerous interesting questions that students may wish to explore. For introductory level students we make the point that a very simple computer model has been quite successful in reproducing the results of their physical experiment, and allows them to perform related experiments that would be awkward to do in the lab. We also emphasize that this is a *dynamical system*, composed of interrelated parts that depend on each other. The output responds to the changing volume in the reservoir, thus producing behavior that was difficult to predict at the beginning of the experiment. Students should appreciate that humans find themselves facing similar difficulties almost any time they attempt to alter the natural environment.

With more advanced students, one might want to address the question of why the model output and experimental data are different, and explore the physics of the cooler that causes the volume of the reservoir to more than double when input is doubled. This, in fact, is precisely how successful models are developed. In our first model, we assumed a linear dependence of output rate on volume. While our results qualitatively match the observations from the real cooler, they do not match quantitatively. This mismatch suggests that we have only partially captured the relevant physics in our assumptions. Thus, we are forced to consider more carefully just how the output rate of the cooler depends on its volume. We find that, according to Bernoulli’s equation for flow through a gravitationally fed pipe, the output rate should depend on the square root of the volume in a vertically walled container (e.g. Sears, Zemansky and Young, 1984, Ch. 13). Our Stella model may be easily modified to incorporate this relationship. The new output term is Output = (8/720)*SQRT(Cooler/8). Now we have a model that not only accurately predicts the behavior of our particular cooler, but does so based on a fundamental understanding of the processes that control its behavior. Thus we can expect to be able to predict the behavior of other, similar systems. This is an excellent example of how the model building process should proceed. Building models in this incremental fashion is a very effective way of developing insight into the system in question.

Many phenomena in natural systems can be usefully approximated with a one-box model with linear output. In an oceanography class students may explore the relationship between marine productivity and nutrient availability with a model similar to the one described above. This model may be easily extended to represent more complex systems (food chain relations etc.). Again, this example will work most effectively if new features are added one at a time, and their impact examined and understood before moving on to the next step.

A one-box model which may be of interest to students of hydrology is the seasonal recharge of an aquifer. The seasonal dependence of the recharge rate may be approximated with a sinusoidal input function (easily accomplished with the SINWAVE function in Stella). Both the reservoir and the output will show sinusoidal variations, but interestingly, they will differ in two ways from the input. The amplitude of the variations will be damped, and the phase of the output variations will lag behind the input (Figure 4). For a given input frequency the amount of damping or attenuation, and the degree of phase shift will depend on the residence time of the aquifer. One can easily...
imagine how this problem can be inverted, so that measurements of rainfall (input) and spring flow (output) can be used to estimate the capacity of a simple aquifer. Another possibility is to add an output for human withdrawal, and investigate the sustainable development of the aquifer. A similar model, based on the phase response of a linear system to periodic input variations, has been used to investigate the response of ocean chemistry to periodic variations in river inputs caused by Pleistocene glaciation (Richter and Turekian, 1993), and periodic recharge of trace elements in a magma chamber (Albarède, 1993).

One of our students' favorite topics is the carbon cycle. In order to develop a realistic simulation of the carbon cycle, a model must include more than one reservoir. We begin with a two-box ocean-atmosphere model. Gradually, students can add a sedimentary rock reservoir, add terrestrial and marine biomass, divide the ocean into shallow and deep reservoirs, and include anthropogenic perturbations. The challenge in this case is to increase the complexity of the model while maintaining a "stable" system, i.e. one that has a non-zero steady state for each of its reservoirs. Students quickly learn the importance of careful balancing of sources, sinks and feedbacks in the carbon system. The mistakes they make in trying to build a stable carbon cycle model are likely to be as instructive as anything else they do.

Conclusions

An increasing amount of geoscience literature reports the results of sophisticated computer models run on giant supercomputers. These models are central to understanding the complex interactions of Earth systems. However there is great utility and pedagogical value in simple systems modeling. All complex models originate simply, and simple models allow for careful scrutiny of the basic physical and mathematical principles behind dynamical behavior. The combination of interactive laboratory experiments with simple computer modeling helps remove the "abstraction" from computer exercises, and allows the student to probe many aspects of the experiment that are not feasible in a lab period due to constraints of time, equipment or facilities. Reconstruction of a system from the ground up allows students to grasp the complex interrelationships of natural systems. They learn that a few basic physical principals can be used to explain a wide variety of natural phenomena, and that by thoughtful application of these principals the students can engage in exercises that allow them to predict the behavior of both simple and complicated systems.

Perhaps the greatest challenge for instructors is to get their students to transcend the desire to simply complete an assignment. We have found that students love to "play" with their computer models. This is one of our goals. We find that our students no longer ask us, "What do I do next?" rather, they ask themselves, "What if I do this?" They appreciate the freedom and creativity that computer model design allows them. Students are empowered by their new-found computer abilities and apply their skills to problems that they find interesting. We believe that by firmly tying their first experience with computer modeling to a physical system that they have previously studied and thus understand well, the students gain confidence in their model and in their abilities. They become excited about discovery and thus about learning, and have taken their first steps toward critical thinking.

References Cited


Software

Stella II, High Performance Systems Inc., 45 Lyme Road, Hanover, NH 03755, Phone (603) 643-9636, Fax (603) 643-9502.

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